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AUTOMATION AND REMOTE CONTROL

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L. S. PONTRYAGIN'S MAXIMUM PRINCIPLE IN OPTIMAL SYSTEM THEORY - II*

L. I. Rozonoér

(Moscow)

Questions are treated which are related to the proofs and applications of L. S. Pontryagin's maximum principle in optimal system theory. The work also contains certain new results. The first portion of the work deals with the optimization problem for the case of a trajectory with a free right end. In the second portion, the maximum principle is formulated for a more general type of boundary conditions. The third part establishes the connection between the method of dynamic programming and the maximum principle, provides a method of solving the optimization problem for linear discrete systems, and also presents a number of considerations in the use of the maximum principle for solving a definite class of problems, related to the theory of dynamic accuracy of control systems.

3. The General Problem

We consider the problem, formulated in Section 1 (Part I) of optimization, given arbitrary boundary conditions for the trajectory's right end. The principle special feature of this problem, as compared with the problem considered in Section 2, is this: that in the present case the requirement that the trajectory's right end lie in set G, imposes, by its very nature, an additional limitation on those controls from among which the optimal one is to be chosen. Indeed, the optimal control must here be chosen only from the class of those controls which guarantee that the system will fall in the set G. Naturally, this circumstance gives rise to additional difficulties. It turns out, however, that the action of the maximum (minimum) principle in this case is completely analogous to the principle formulated in Section 2, and the special feature of the problem under consideration here, as cited above, is expressed in the greater complexity of the boundary conditions for the corresponding differential equations. We first give the fundamental result (theorem 3) which asserts the applicability of the maximum principle, and we then give a method of determining the initial conditions.

In the sequel, we shall always assume that the set G in which the end of the trajectory x(t) must lie is closed and convex**.

In the proof of the fundamental theorem (Cf., the Appendix), it is also assumed that the set G has at least one interior point in phase space X, i.e., a point in a sufficiently small neighborhood of which there are contained no points which do not lie in G. It is easily proven that, with this, there are interior points in any arbitrarily small neighborhood of any point of G.

a. The Maximum Principle

We now turn to the exposition of the basic result. Let the initial system of equations have the form:

$$\dot{x}_i = f_i(x_1, \ldots, x_n;\ u_1, \ldots, u_r;\ t),\ x_i(T_0) = x_i^0 \quad (i = 1, \ldots, n).$$

*Part I of this work was published in Automation and Remote Control (USSR) 20, 10 (1959).

** A set G is said to be convex if the straight line-segment joining any two points of G, lies completely in the set G.

We consider the set of vector functions p(t) which satisfy the relationships

$$\dot{p}_i = -\sum_1^n p_s \frac{\partial f_s(x, u, t)}{\partial x_i} \qquad (i = 1, \ldots, n). \tag{2}$$

Equation (2) (in contradistinction to the problem of Section 2) does not define the functions $p_i(t)$, since the boundary conditions for p(t) are still not fixed. Therefore, even if the control u(t) and the trajectory $x^u(t)$ corresponding to u(t) are given, the vector p(t) is still not uniquely defined by (2).

As before, we introduce the function $H(x, p, u, t) \equiv \sum^n p_i f_i(x_1, \ldots, x_n; u_1, \ldots, u_r, t)$, by means of which Eq. (1) and Eq. (2) are written in the form

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad x_i(T_0) = x_i^0 \quad (i = 1, \ldots, n). \tag{3}$$

Let u(t) be some admissible control, $x^u(t)$ the trajectory corresponding to it, and p(t) some one of the vector functions which satisfy relationships (2). By substituting the values of the variables $x_i^u(t)$, $p_i(t)$ in the function H (x, p, u, t), we obtain the quantity

$$K(t, u_1, \ldots, u_r) \equiv H[x(t), p(t), u, t],$$

which, for each fixed moment of time t, is a function of the point $u = (u_1, \ldots, u_r)$ which lies in the set U of the space R. We say that control u(t) satisfies the maximum (minimum) condition with respect to the vector p(t) if, at any fixed moment of time t ($T_0 \leq t \leq T$), the function K(t, u) attains an absolute maximum (minimum) on the set U for given variables, equal to the values of the control at the same moment of time, i.e., for $u_k = u_k(t)$ ($k = 1, \ldots, r$).

As was shown in Section 2, Part I, the geometric meaning of the maximum (minimum) condition amounts to this: that the control which satisfies this condition tends, at each moment of time, to "accelerate" maximally the representative point in the direction defined by the vector p(t).

We again emphasize that vector p(t) is, on the one hand, not arbitrary and, on the other hand, is still not fixed (in contradistinction to the problem of Section 2, Part I). Therefore, if some permissible control u(t) is given, it is not immediately clear whether one may choose such a vector p(t) that, relative to this p(t), the control u(t) would satisfy the maximum (minimum) conditions. It turns out that an optimal control possesses precisely this property. This fact also constitutes the basic content of L. S. Pontryagin's maximum principle.

In order to formulate exactly the theorem which expresses the maximum principle, we make one more remark. We consider the function $\psi(x) \equiv \sum^n c_i x_i$ of points $x = (x_1, \ldots, x_n)$ on the set G. The set of points $x^* (x^*_1, \ldots, x^*_n)$, at which the function $\psi(x)$ assumes a value which is a minimum with respect to all other points of G, we denote by G^*. If the set G^* exists, then the following inequality always holds: $\psi^* \equiv \sum_1^n c_i x_i^* \leqslant \sum_1^n c_i x_i$ where x^* is an arbitrary point for G^* and x is an arbitrary point of G). The functional $S = \sum^n c_i x_i(T)$ can not, obviously, assume values less than ψ^*. Then, an arbitrary control which takes the system over into set G^* is min-optimal, since for such a control the functional assumes the least of all its possible values, equal to ψ^*. Therefore, if the time, $T - T_0$, during which the system passes in the necessary optimal way to set G, is given in such a way that the set G^* turns out to be admissible, the problem of minimizing the functional S then reduces to the problem of determining some control which takes the system from the fixed point x^0 to the fixed set G^*. We call this a degenerate problem. It is not considered in the present paper.

Analogously, we define the case when one deals with a maximum of the function $S = \sum_1^n c_i x_i(T)$. as a degenerate.

The nondegeneracy of the problem may be perceived in practice by comparing the given time $T-T_0$ with the minimal time T^* necessary for translation of the system from point x^0 to the set G^*.

We note further that the problem considered in Section 2, Part I can not be degenerate for any given time T. Indeed, in the given case, the function $\sum_1^n c_i x_i$ is not limited to the set G (occupying the entire space) and, consequently, assumes an extremal value at no point. The matter is analogous in those cases when the set G is open* or is not bounded.

We now formulate the fundamental theorem.

Theorem 3. If, in a nondegenerate problem, the control u(t) is min-optimal (max-optimal) with respect to $S = \sum_1^n c_i x_i(T)$, then there exists a vector function $p^u(t)$ such that u(t) satisfies the maximum (minimum) condition with respect to $p^u(t)$.

Theorem 3 is valid, certainly, in that special case which was considered in Section 2, Part I. Theorem 1 asserts that, in the case of the problem with a free trajectory right end, the vector $p^u(t)$, the existence of which is guaranteed by Theorem 3, is subject to the boundary condition $p^u(T) = -c$. It is now necessary to formulate an analogous condition for the vector $p^u(t)$** in the general problem being considered.

Theorem 3 and the boundary conditions formulated below can be obtained on the basis of the results of work [1], in which the maximum principle with the corresponding transversality conditions was proven under more general assumptions.

b. The Boundary Conditions

Let the point $x^1 = (x_1^1, \ldots, x_n^1)$ be the end of an optimal trajectory x(t). That portion of set G for which $\sum_1^n c_i x_i \leqslant \sum_1^n c_i x_i^1$, we denote by G^-; the portion for which $\sum_1^n c_i x_i \geqslant \sum_1^n c_i x_i^1$, we denote by G^+. The sets G^- and G^+ obviously, are closed and convex and have the hyperplane C, given by the equation $\sum_1^n c_i (x_i - x_i^1) = 0$, for their common boundary. Point x^1, lying in both G^- and G^+, is a boundary point of both sets. For definiteness, we shall now consider the case when one seeks a minimum of the functional $S = \sum_1^n c_i x_i(T)$.

If the control u(t) is min-optimal, then there do not exist controls from the admissible class which would, during the time $T-T_0$, translate the system into set G^- since, for interior points of set G^-, the value of the quantity $\sum_1^n c_i x_i$ is less than for the point x^1. With this, it turns out (Cf., the Appendix) that the vector p(T), at the moment of time t = T, must be orthogonal to some hyperplane A, which is a bracket*** for set G^- at the point x^1 and, moreover, must be directed to that side of hyperplane A where set G^- lies. Therefore, if the equation of

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* A set G is said to be open if none of the boundary points of set G lies in the set.

** In the sequel, the superscript u will be omitted from $p^u(t)$.

*** A hyperplane A, containing the point x^1, is called a bracket to the convex set G^- at the point x^1 if set G^- lies completely to "one side" of A. It has been proven that a bracket hyperplane may be passed through each boundary point of a convex set.

hyperplane A has the form $\sum_1^n a_i(x_i - x_i^1) = 0$, and the signs of the coefficients a_i are so chosen that set G^- lies on the side of the hyperplane where $\sum_1^n a_i(x_i - x_i^1) \leqslant 0$, then

$$p_i(T) = -a_i \quad (i = 1, \ldots, n) \tag{4}$$

or $p(T) = -a$, if we introduce the vector $a = (a_1, \ldots, a_n)$. Equation (4) generalizes the corresponding conditions for the problem of Section 2, Part I. In fact, two cases of the placement of point $x^1 = x(T)$ are possible: either inside G or on the boundary of G. In the first case (Fig. 1) there may be passed the unique hyperplane A, bracketing G^- and coinciding with hyperplane C, through point x^1. Any other hyperplane A through point x^1 will necessarily "intersect" the set G^- and, consequently, will not be a bracket. Therefore, if the end of the trajectory lies in the interior of set G, the $a_i = c_i$ and $p_i(T) = -c_i$. In the problem considered in Section 2, Part I, this case was always realized, since the set G occupied the entire space and in general, did not have boundary points*.

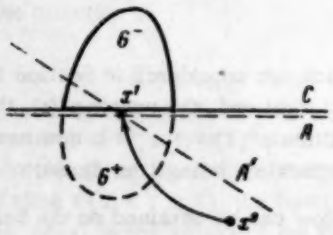

Fig. 1

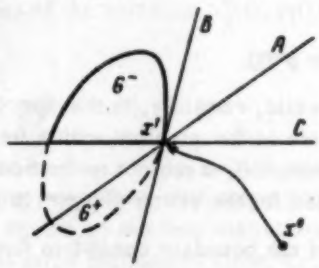

Fig. 2

If the second case holds, and point x^1 is a boundary point of set G, then the bracket hyperplane to G^- is not uniquely defined (Fig. 2). We shall assume that, at point x^1, there exists a unique hyperplane B which is a bracket to set G. The sign of the coefficients b_i of the hyperplane B are so chosen that set G lies on that side of B where $\sum_1^n b_i(x_i - x_i^1) \leqslant 0$. It is then easily seen that no hyperplane A which is a bracket to set G^- can lie in that region of space where the inequalities $\sum_1^n c_i(x_i - x_i^1) \leqslant 0$ and $\sum_1^n b_i(x_i - x_i^1) \leqslant 0$ hold simultaneously. By now taking into account that G^- lies on that side of A where $\sum_1^n a_i(x_i - x_i^1) \leqslant 0$, we obtain the following expression for the coefficients a_i:

$$a_i = \lambda c_i + \mu b_i \quad (i = 1, \ldots, n), \tag{5}$$

where $\lambda \geq 0$ and $\mu \geq 0$ are certain nonnegative numbers which do not reduce to zero simultaneously. For $\lambda = 0$, hyperplane A coincides with B and, for $\mu = 0$, with C. If $\lambda > 0$ (or $\mu > 0$) we can, without loss of generality, set $\lambda = 1$ (or $\mu = 1$). Of both constants are positive, then either of them can be set equal to unity.

This follows from the fact that, to define hyperplane A, the coefficients a_i need only be given to within an arbitrary positive constant (this constant is required to be positive in order that there be no change in the sense of the inequality $\sum_1^n a_i(x_i - x_i^1) \leqslant 0$). Therefore, essentially, only one constant enters into conditions (5).

* This case is also always realized in the problem in which the set G is open. For these problems, all the results of Section 2, Part I are completely valid.

Let the set G be given by the inequality $F(x_1, \ldots, x_n) \leq 0$. Then the coordinates of the boundary points of set G (including the coordinates of point x^1) must satisfy the condition

$$F(x_1, \ldots, x_n) = 0. \tag{6}$$

For each boundary point we can find coefficients $b_i(x_1, \ldots, x_n)$ which define the hyperplane B, which is a bracket to G at the point $\underline{x}$. In particular, if the function $F(x_1, \ldots, x_n)$ is differentiable, then the bracket is the hyperplane which is tangent to surface (6) and therefore

$$b_i(x_1, \ldots, x_n) = \frac{\partial F(x_1, \ldots, x_n)}{\partial x_i} \quad (i = 1, \ldots, n).$$

By uniting conditions (4)-(6), we convince ourselves that the coordinates of vectors p(t) and x(t) at time t = T must satisfy the set of relationships

$$\begin{gathered} p_i(t) = -\lambda c_i - \mu b_i[x_1(T), \ldots, x_n(T)] \quad (i = 1, \ldots, n), \\ F[x_1(T), \ldots, x_n(T)] = 0. \end{gathered} \tag{7}$$

We thus obtain n + 1 equations, (7), in which, as was stated above, one additional constant (λ or μ) must enter. Therefore, (7) gives essentially $\underline{n}$ independent relationships. Together with the $\underline{n}$ relationships

$$x_i(T_0) = x_i^0 \quad (i = 1, \ldots, n) \tag{8}$$

we obtain 2n boundary conditions for the solution of system (3) of differential equations, also of order 2n. The control u(t) is eliminated from system (3) by the same method as in the case of the problem of Section 2, since the maximum condition allows one, generally speaking, to express u(t) at each moment of time in terms of x(t) and p(t).

In the case when functional $S = \sum_1^n c_i x_i(T)$, attains a maximum, boundary conditions (7) and (8) are retained if the inequality giving the set G is written in the form $F(x_1, \ldots, x_n) \geq 0$. To eliminate u(t) from system (3), it is necessary to use the minimum condition.

c. Determination of the Boundary Conditions in Certain Particular Cases

If some coordinate x_s does not enter into the inequality which defines the set G, then the corresponding coefficient b_s in (7) is equal to zero (G is a "cylindrical" body with its generator parallel to the x_s axis and, consequently, the bracketing hyperplane is also parallel to the x_s axis).

In many cases of practical importance the set G lies on some linear manifold of phase space X and, consequently has no interior points in X (since in any arbitrarily small neighborhood of any point of a linear manifold one can find a point which does not lie in the manifold). In particular, this circumstance occurs if some of the coordinates are exactly fixed at time t = T but the set G is given by the system of equations

$$x_s(T) = x_s^1 \quad (s = 1, \ldots, q;\ q < n). \tag{9}$$

We consider this case. Since, in practice, a sufficiently small violation of conditions (9) is always admitted, one could require that instead of (9), the following inequality hold

$$\sum_1^q (x_s - x_s^1)^2 \leqslant \epsilon^2, \tag{9'}$$

which, for small ϵ, guarantees that the violation of conditions (9) is sufficiently small and that $|x_s - x_s^1| \leq \epsilon$. Inequality (9') gives a convex and closed domain with interior points in X. However, it turns out to be possible

to write simpler boundary conditions than those which follow from (9'). Conditions (9) can be directly considered as the boundary conditions for system (3) by replacing by them the conditions from (7) for the corresponding q. Since, with this, the equations defining G do not depend on the remaining n−q coordinates, then $b_{q+1} = \ldots = b_n = 0$ so that there remains in the result the following set of 2n boundary conditions:

$$\begin{aligned} x_i(T_0) &= x_i^0, \quad i = 1, \ldots, n, \\ x_s(T) &= x_s^1, \quad s = 1, \ldots, q, \\ p_m(T) &= -c_m, \; m = q+1, \ldots, n. \end{aligned} \tag{10}$$

We now consider the frequently arising problem when it is required to determine the control which gives an extremum to the integral $\int_{T_0}^{T} \Phi(x, u)\, dt$, where, for $t = T_0$ and $t = T$, the values of the coordinates must be fixed:

$$x_i(T_0) = x_i^0, \quad x_i(T) = x_i^1 \quad (i = 1, \ldots, n).$$

The new variable $x_{n+1}(t) = \int_{T_0}^{t} \Phi(x, u)\, dt$ does not enter either into definition of set G or in the right members of the original system of differential equations. By virtue of this latter fact, $p_{n+1}(t) = 0$. Since $c_1 = c_2 = \ldots = c_n = 0$, c_{n+1} and $b_{n+1} = 0$, then $p_{n+1}(t) \equiv p_{n+1}(T) = -1$. The system of differential equations and the boundary conditions for them take the following form:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad H \equiv \sum_1^n p_s f_s(x, u, t) - \Phi(x, u), \tag{11}$$

$$x_i(T_0) = x_i^0, \; x_i(T) = x_i^1 \quad (i = 1, \ldots, n).$$

The equation $\dot{x}_{n+1} = \Phi(x, u)$ is excluded, since it gives no additional condition.

d. Optimization Problems in Which the Time of Motion is not Fixed Beforehand

Until now we have considered problems where the time required to implement the optimal process in the system was fixed. On the basis of the results obtained, we can now easily formulate the solution for the problems in which the amount of time taken by the system to transfer from point x^0 to the set G is given beforehand, but must be so chosen that the functional takes a least (greatest) value in comparison with the values which would be obtained for all other possible amounts of this time.

We note first of all that if the problem is degenerate for at least some values of T, then these values of T also provide a minimal (maximal) value of the functional in comparison with all other values, such that the optimization problem with nonfixed time reduces to the degenerate problem. We shall therefore assume that the problem, with arbitrary fixed values of T, is nondegenerate.

It is obvious that all the results given in this section must remain in force for problems with "free" times. Indeed, if the time sought exists and is equal to $\overline{T}$, one can consider the problem with fixed time $T = \overline{T}$, the solution of which would be the solution of the initial problem. However, since the magnitude of $\overline{T}$ is unknown, one additional condition which would define $\overline{T}$ is still required. It turns out that such a condition is the requirement in accordance with which the function H(x,p,u,t) must equal zero at the end of the motion:

$$\sum_1^n p_i(T) f_i[x(T), u(T), T] = 0.$$

If u(T), in correspondence with the maximum (minimum) condition, is expressed in terms of x(T) and p(T) and substituted in (12), the equation obtained gives the additional boundary condition which, in conjunction with boundary conditions (7), allows one to find the solution of system (3) and the optimal time $\overline{T}$.

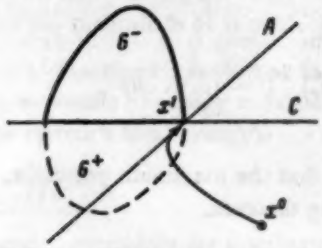

Fig. 3

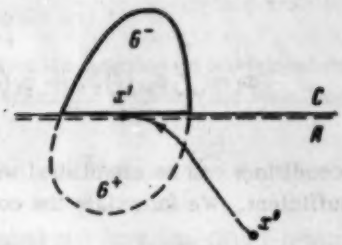

Fig. 4

Condition (12) has a simple geometric meaning. Indeed, the function H is the scalar product of the vector $p(t)$ by the vector $\dot{x}(t)$ for the velocity of the representative point. The scalar product equalling zero means that at the end of the motion the velocity vector must be orthogonal to $p(T)$ and, consequently, must (for example, in the min-optimal case) lie in hyperplane A, which is a bracket for set G^- (Fig. 3). This requirement becomes particularly clear in the case when the end of the optimal trajectory lies in the interior of G (Fig. 4). If the velocity vector were directed to the interior of G^- then, by increasing the time T, one could so arrange matters that the end of the trajectory would lie inside of G^- which is impossible if the control u(t) is min-optimal. If the vector $\dot{x}(T)$ were directed from G^- "outward", this would mean that for a smaller value of T the representative point would be still found interior to G^- which, again is impossible.

e. Problems of Maximal Speed of Operation

We can now turn to the problem of the minimum time of transition from the given point x^0 to the set G. For its solution, it suffices to add the equation $\dot{x}_{n+1} = \varkappa$ (where $\varkappa$ is an arbitrary positive constant) and to solve the problem of minimizing $x_{n+1}(T)$ for a time not fixed beforehand. It is easily seen that with this we have in boundary condition (7), $c_1 = \ldots = c_n = 0$, $p_{n+1}(T) = -1$ and, in (12) $H \equiv \sum_1^n p_i(T) f_i[x(T), u(T), T] - \alpha = 0$. Moreover, $\dot{p}_{n+1}(t) = 0$ and $p_{n+1}(t) \equiv -1$. Since α is an arbitrary positive constant, we can introduce the function $\overline{H} \equiv \sum_1^n p_i f_i(x, u, t)$, equal to $H + \alpha$ and, finally, we can write the system of differential equations in the form:

$$\dot{x}_i = \frac{\partial \overline{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \overline{H}}{\partial x_i} \quad (i = 1, \ldots, n) \tag{13}$$

and its boundary conditions in the form

$$x_i(T_0) = x_i^0, \quad p_i(T) = -\mu b_i[x_1(T), \ldots, x_n(T)] \quad (i = 1, \ldots, n),$$

$$F[x_1(T), \ldots, x_n(T)] = 0, \quad \sum_1^n p_s(T) f_s[x(T), u(T), T] > 0. \tag{14}$$

If the set G is a fixed point x^1 then, by substituting $\Phi(x, u) \equiv \alpha > 0$ in formula (11), we easily obtain the following boundary conditions for system (13):

$$x_i(T_0) = x_i^0, \quad x_i(T) = x_i^1 (i = 1, \ldots, n), \quad \sum_1^n p_s(T) f_s[x(T), u(T), T] > 0. \tag{15}$$

f. Linear Systems

The maximum principle, as was already mentioned for the case of the particular problem of Section 2,

gives, in general, only necessary conditions for optimality. However, for linear systems, described by equations of the form

$$\dot{x}_i = \sum_1^n a_{ik}(t)\, x_k + \varphi_i(u_1, \ldots, u_r) \quad (i = 1, \ldots, n), \tag{16}$$

sufficient optimality conditions can be established which show that the maximum principle, as applied to such systems, is "almost" sufficient. We formulate the corresponding theorem.

As before, we will denote by $b_i(x_1^1, \ldots, x_n^1)$ the coefficients of the hyperplane bracketing set G at the point $x^1 = (x_1^1, \ldots, x_n^1)$ if point x^1 is a boundary point of G. To solve the problem of a minimum of functional S, the signs of the coefficients b_i are so chosen that, for each point $x(x_1, \ldots, x_n) \in G$, the inequality $\sum_1^n b_i(x_1^1, \ldots, x_n^1) \cdot (x_i - x_i^1) \leq 0$, holds, and for solving the problem of a maximum of the functional, the inequality holds with reversed sense. We do not assume now that there is a unique bracketing hyperplane to G at the point x^1, so that we denote by $b_i(x_1^1, \ldots, x_n^1)$ the coefficients of any bracketing hyperplane.

If point x^1 is interior to G, we should set $b_i(x_1^1, \ldots, x_n^1) = 0$. Then, in the case of a minimum of the functional, the following inequality always holds for $x^1 \in G$ and $x \in G$, $\sum_1^n b_i(x_1^1, \ldots, x_n^1)(x_i - x_i^1) \leqslant 0$ (and the inequality with reversed sense for the problem of the maximum of the functional).

Now let x(t) be the trajectory corresponding to control u(T), and let $x(T) = x^1$. Then the following theorem holds*.

Theorem 4. If in system (16), control u(t) satisfies the maximum (minimum) condition with respect to vector p(t), whose coordinates at time t = T take the value,

$$p_i(T) = -\lambda c_i - \mu b_i(x_1^1, \ldots, x_n^1) \quad (i = 1, \ldots, n),$$

where μ is nonnegative and λ is positive, then the control is min-optimal (max-optimal) for $S = \sum_1^n c_i x_i(T)$.

The conditions of Theorem 4 are very "close" to the necessary conditions for optimality expressed by Theorem 3 and formulas (4) and (5) of this section. The sole difference amounts to this that, unlike formulas (4), it is required in the conditions for the theorem just formulated that the constant λ be strictly positive.

The proof of Theorem 4 is given in Appendix II, Section 2*. We remark that the proof rests solely on Theorem 2, on the sufficiency of the maximum principle in linear problems with free-trajectory right ends and, in particular, does not assume that the set G must have interior points.

g. Continuity and Piecewise Differentiability of the Function H

We now formulate a theorem which expresses a very characteristic property of optimal processes. Let the control u(t) be given on the segment $[T_0, T]$ and let x(t) satisfy system (1). Then the following theorem is valid.

Theorem 5. If control u(t) satisfies the maximum (minimum) condition with respect to p(t) then the function $M(t) \equiv H[x(t), p(t), u(t), t]$ is continuous and piecewise-differentiable on the interval (T_0, T) where, at the continuity points of control u(t) the derivative exists and equals

$$\frac{dM}{dt} = \frac{\partial H[x(t), p(t), u(t), t]}{\partial t}.$$

It follows from Theorem 5 that, if the right members of differential Equations (1) do not explicitly depend on time, and if $\frac{\partial H}{\partial t} = \sum_1^n p_i \frac{\partial f_i(x, u)}{\partial t} = 0$, then $\dot{M} = 0$ and (with continuity taken into account) the function

* In the problems where time T is not fixed, the theorem analogous to Theorem 4 apparently does not hold.

H[x(t), p(t), u(t)] is constant. If, in particular, H[x(t), p(t), u(t)] reduces to zero at time t = T as, for example, in formula (12), then the magnitude of H is identically zero for any t.

The constancy of function H, analogous to the Hamiltonian function of analytical mechanics, corresponds to the law of the conservation of energy in conservative mechanical systems (the Hamiltonian function, as is well known, expresses the system's total energy).

h. Singular Controls

The recommended procedure for eliminating the control u(t) from Eqs. (3) by means of the maximum (minimum) condition may turn out to be impossible if the function H, for definite values of x and p, does not in fact depend on u. This may easily be followed in the simplest example of determining the control which, during fixed time T, takes the system described by the equations

$$\dot{x} = u, \quad |u| \leqslant M,$$

from point $x(0) = x^0$ to point $x(T) = x^1$ and minimizes the integral $^1/_2 \int_0^T x^2 dt$. In correspondence with formulas (11), we can write

$$H \equiv pu - \frac{1}{2}x^2, \quad \dot{p} = x.$$

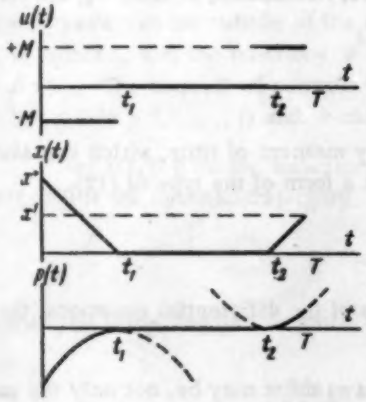

Fig. 5

It follows from the maximum condition that, for $p \neq 0$, the control equals u = M sign p. However, p = 0, the function H does not depend on u. In addition, as may easily be seen, the optimal control, coordinates and pulses, for the condition $T > (x^0 + x^1)/M$, must be changed as shown by Fig. 5. Therefore, on the time interval $[t_1, t_2]$, the function $p(t) \equiv 0$, and the control u(t) is not defined by the maximum condition. Such a control can be called singular. A singular control, if such exists, can be easily detected. Indeed, by first determining for which values of x and p (in the given example, for p = 0) H does not depend on u, we can seek such a control for which this independence would subsist in time. In the given example (since p = 0 = const), $\dot{p} = 0$ and, consequently, x = 0, from which we obtain the singular control u = 0. The function p(t) and, consequently, the control u(t), on the entire segment [0, T], can be defined by the condition of continuity of the singular solution with the nonsingular solutions (in ghe given example, the solution for u = 0 with the solutions for u = ± M). Theorem 5 can also be used. In the given case, for the singular control H = 0 and, for the nonsingular control, $H = M|p| - x^2/2$. Since H = const = 0, we have

$$|p(0)| = \frac{1}{2M}(x^0)^2, \quad |p(T)| = \frac{1}{2M}(x^1)^2,$$

and the signs of the quantities p(0) and p(T) are quickly determined.

i. The Synthesis Problem

We now consider the methods of using the practice the maximum principle of L. S. Pontryagin for solving the problem of designing optimal systems.

The problem of constructing optimal systems may be posed in two aspects. In the first case it may be sufficient to calculate beforehand an optimal control in the form of a function of time for definite, fixed and previously given initial conditions, and thereafter to design a program-controlled system. In the second case, the system's initial conditions may be unknown beforehand, but, nevertheless, it is required that the system realize an

optimal process with arbitrary initial conditions. In other words, in this case, the optimal control must be expressed in the form of a function of the current value of the system coordinate and, possibly, of time, $u = \varphi(x, t)$:

$$u_k = \varphi_k(x_1, \ldots, x_n, t) \quad (k = 1, \ldots, r), \tag{17}$$

which allows one, by the use of a regulator with nonlinear connections, to design a system which is optimal for any initial conditions. This problem bears the name of the optimal-system-synthesis problem.

The maximum principle essentially solves the first problem immediately, since the finding of the optimal control in the form of a function of time amounts to the integration of a definite system of differential equations with the corresponding boundary conditions. It should be mentioned, however, that the technical difficulties of integrating the differential equations which flow from the maximum principle require, as a reule that computer technology be called into play. Integration of the equations in closed form turns out to be possible, naturally, in rare cases only.

The synthesis problem ordinarily arises in those problems in which the time of motion is not fixed beforehand. The following procedure for determing the functions (17) derives from the maximum principle. From the maximum (minimum) condition one determines the control in the form of the function $u = \psi(x, p, t)$. With fixed initial conditions, $x(T_0) = x^0$, there are given various initial values of $p(T_0) = p^0$, so chosen that the solution of system (3) of equations, with initial conditions $x(T_0) = x^0$ and $p(T_0) = p^0$, is satisfied together with the required boundary conditions. If the value of p^0 is not thereby defined uniquely, one then chooses the set of values, $p_i^0 (i = 1, \ldots, n)$, for which the functional's value turns out to be extremal. For each fixed set of values of $x_i^0 (i = 1, \ldots, n)$ at the fixed moment of time T_0, there is thus obtained a single set of values of $p_i^0 (i = 1, \ldots, n)$ which provide the optimal process. The function $p^0 = \chi(x^0, T_0)$ is thus defined. Therefore, at time T_0, the following equation holds

$$u(T_0, x^0) = \psi[x^0, \chi(x^0, T_0), T_0].$$

But it is easily seen that this relationship must also hold at any arbitrary moment of time, which can always be taken as the initial moment, by virtue of which the control u is defined in a form of the type of (17):

$$u = \psi[x, \chi(x, t), t].$$

In those cases when time does not enter explicitly in the right members of the differential equations, the function φ_k in (17) also turn out to be independent of time.

We now turn our attention to the circumstance that among the variables x_i there may be, not only the coordinates of the controlled object, but also artificially introduced quantities (for example, in the case of an integral extremum, the quantity $x_{n+1}(t) = \int_{T_0}^{t} \Phi(x, u)\, dt$). However, in those cases when the coordinates x_s does not enter explicitly into the right members of the differential equations or into the relationships defining the domain G, the quantity $p_s(t)$ is constant in time and does not depend on the boundary values of the coordinates $x_1, \ldots, x_n$. In addition, x_s does not enter into the function H and, consequently, does not enter into the expression for the control u in terms of x, p, and t. Therefore, relationship (17) will also not contain x_s. In particular, in the case of the optimization of the integral $x_{n+1}(T) = \int_{T_0}^{T} \Phi(x, u)\, dt$, only the object's coordinates, and not the quantity x_{n+1}, will enter into (17).

The synthesis problem becomes much more complicated if the artificially introduced coordinate, although not entering explicitly in the equations' right members, does figure in the relationships which define region G. Examples of this are the cases of "integral limitations" of the type $\int_{T_0}^{T} \Phi(x, u)\, dt \leqslant A$, imposed on the control. Here, the introduced parameter $x_{n+1}(t) = \int_{T_0}^{t} \Phi(x, u)\, dt$ enters into the definition of region G: $\dot{x}_{n+1}(t) \leq A$. Therefore, functions (17) generally turn out to depend on x_{n+1}. In this case, in order to realize an optimal

system, one must either have a computing device which determines $x_{n+1}(t)$ at each moment of time, or one must previously compute x_{n+1} in the form of a function of the current, $x_i (i = 1, \ldots, n)$, and initial, x_i^0, coordinate values.

The synthesis problem, to an even greater degree than the problem of determining the control in the form of a function of time, requires the use of computing technology. Even in those cases when it is possible to determine functions (17) in closed form, the form of these functions is ordinarily very complicated. Therefore, a foremost role is played by the problem of synthesizing systems which are sufficiently simple and, in addition, are close to optimal.

Another synthesis procedure, based on the solution of partial differential equations, derives from the dynamic programming method of Richard Bellman.

We will speak of this in the following part of this work [Cf., Automation and Remote Control (USSR) 20, 12 (1959)].

APPENDIX

1°. The Proof of Theorem 3

We define a class Γ of modified controls of a special form. On the interval (T_0, T), we arbitrarily choose $j \geq 1$ moments of time $t = t_k$ and we construct the system of j nonintersecting segments $I_k = [t_k - \tau, t_k]$ of length τ, where the number $\tau > 0$ is sufficiently small ($k = 1, \ldots, j$). We introduce j r-dimensional piecewise-continuous vector functions $\varphi^k(z) \equiv [\varphi_1^k(z), \ldots, \varphi_r^k(z)]$, defined on the segment $[0, 1]$, on $\varphi^k(z) \in U$ $(k = 1, \ldots, j)$.

In addition, to control $u(t)$ we consider the modified control $v(t; \tau, t_1, \ldots, t_j; \varphi^1, \ldots, \varphi^j)$ such that $v = u(t)$ everywhere outside of the segments I_k and $v = \varphi^k[(t_k - t)/\tau]$ on I_k. Control $v(t)$ is obviously admissible. The points t_k and the functions $\varphi^k(z)$ are called, respectively, defining points and defining functions of the control $v(t)$. The class Γ of controls includes all the controls $v(t; \tau, t_1, \ldots, t_j; \varphi^1, \ldots, \varphi^j)$ with arbitrary j, t_k, $\varphi^k(z) \in U (k = 1, \ldots, j)$ and τ sufficiently small.

Let $x(t, [v])$ be the trajectory corresponding to control $\in \Gamma$ and let $\delta x(t, [v]) = x(t, [v]) - x(t, [u])$, where $x(t, [u])$ is the optimal trajectory. For $x(t, [v])$ we have the system of equations

$$\delta \dot{x}_i = f_i(x + \delta x, v) - f_i(x, u) \quad (i = 1, \ldots, n), \tag{18}$$

which, by using Taylor's formula, we can rewrite in the form

$$\delta \dot{x}_i = \sum_1^n \frac{\partial f_i(x, u)}{\partial x_s} \delta x_s + f_i(x, v) - f_i(x, u) + \alpha_i(t, [v]) \quad (i = 1, \ldots, n), \tag{19}$$

where

$$\alpha_i(t, [v]) = \sum_1^n \left[\frac{\partial f_i(x, v)}{\partial x_s} - \frac{\partial f_i(x, u)}{\partial x_s} \right] \delta x_s + \frac{1}{2} \sum_{s, q=1}^n \frac{\partial^2 f_i(x + \vartheta \delta x, v)}{\partial x_s \partial x_q} \delta x_s \delta x_q \tag{20}$$

and $0 < \vartheta < 1$. By integrating (19) between the limits of T_0 and $t (t_m < t < t_{m+1} - \tau)$, we get

$$\delta x_i(t, [v]) = \int_{T_0}^t \sum_1^n \frac{\partial f_i(x, u)}{\partial x_s} \delta x_s \, dt + \sum_{k=1}^m J_i^k([v]) + \beta_i(t, [v]); \; t \in (t_m, t_{m+1} - \tau) \tag{21}$$

where

$$J_i^k([v]) = \int_{t_k - \tau}^{t_k} \left[f_i\left(x(t), \varphi^k\left(\frac{t_k - t}{\tau}\right)\right) - f_i(x(t), u(t)) \right] dt,$$

$$\beta_i(t, [v]) \equiv \int_{T_0}^{t} \alpha(t, [v])\,dt = \frac{1}{2}\int_{T_0}^{t} \sum_{s,q=1}^{n} \frac{\partial^2 f_i(x + \vartheta\delta x, v)}{\partial x_s \partial x_q}\,\delta x_s \delta x_q\,dt +$$

$$+ \sum_{k=1}^{m} \int_{t_k-\tau}^{t_k} \sum_{1}^{n} \left[\frac{\partial f_i\left(x, \varphi\left(\frac{t_k - t}{\tau}\right)\right)}{\partial x_s} - \frac{\partial f_i(x, u)}{\partial x_s} \right] \delta x_s\,dt.$$

By taking into account the boundedness of $\partial f_i / \partial x_s$ and $\partial^2 f_i / \partial x_s \partial x_q$ on $[T_0, T]$ and also the estimate for δx_s

$$|\delta x_s(t, [v])| \leqslant M\tau, \quad t \in [T_0, T]$$

(where the constant M does not depend on τ), we easily obtain the estimate for β_i:

$$|\beta_i(t, [v])| \leqslant N\tau^2, \quad t \in [T_0, T],$$

where N also does not depend on t and τ, by virtue of which, uniformly in t,

$$\lim_{\tau \to 0} \frac{\beta_i(t, [v])}{\tau} = 0. \tag{22}$$

We now consider the integrals $J_i^k([v])$ (21). If we make the change of variables $z = (t_k - t)/\tau$, we shall have

$$J_i^k([v]) = \tau \int_0^1 [f_i(x(t_k - \tau z), \varphi^k(z)) - f_i(x(t_k - \tau z), u(t_k - \tau z))]\,dz.$$

As $\tau \to 0$, by virtue of the continuity of x(t) and the continuity of u(t) to the left, i.e., $u(t-0) = u(t)$, we get

$$R_i(t_k, [\varphi^k]) \equiv \lim_{\tau \to 0} \frac{J_i^k([v])}{\tau} = \int_0^1 [f_i(x(t_k), \varphi^k(z)) - f_i(x(t_k), u(t_k))]\,dz. \tag{23}$$

It is easily shown that the limit $y_i(t[v]) \equiv \lim_{\tau \to 0} \delta x_i(t, [v])/\tau$ $(i = 1, \ldots, n)$, which we call the variation of trajectory x(t) at the point t, exists for all $t \in [T_0, T]$ different from t_k. Then, by dividing (21) by τ and by taking (22) and (23) into account, we get

$$y_i(t, [v]) = \int_{T_0}^{t} \sum_{1}^{n} \frac{\partial f_i(x, u)}{\partial x_s} y_s\,dt + \sum_{k=1}^{m} R_i(t_k, [\varphi^k]), \quad t \in (t_m, t_{m+1}). \tag{24}$$

It follows from (24) that, on the intervals (t_m, t_{m+1}), the functions $y_i(t[v])$ satisfy the system of differential equations in the variations

$$\dot{y}_i = \sum_{1}^{n} \frac{\partial f_i(x, u, t)}{\partial x_s} y_s \quad (i = 1, \ldots, n), \tag{25}$$

and, at the points $t = t_k$, experience the jumps

$$y_i(t_k + 0) - y_i(t_k - 0) = R_i(t_k, [\varphi^k]) \quad (i = 1, \ldots, n;\ k = 1, \ldots, j), \tag{26}$$

where, obviously, $y_i(t_1-0) = 0$ (since $\delta x_i(t[v]) = 0$ for $t \in [T_0, t_1 - \tau]$). We denote by $y_i^k(t)\,(t > t_k)$ the solutions of system (25) with the initial conditions $y_i(t_k) = R_i(t_k, [\varphi^k])$. Due to the linearity of the equations, the solution of (25) with conditions (26) is obtained in the form

$$y_i(t, [v]) = \sum_{k=1}^{m} y_i^k(t), \quad t \in (t_m, t_{m+1}). \tag{27}$$

We now determine the $y_i^k(t)$. If $\zeta^q(t) = [\zeta_1^q(t), \ldots, \zeta_n^q(t)]\,(q = 1, \ldots, n)$ is a fundamental system of solutions of (25), and the matrix $\| \vartheta_i^q(t) \|$ is the inverse of $\| \zeta_i^q(t) \|$, then

$$y_i^k(t) = \sum_{s,\,q=1}^{n} \zeta_i^q(t)\, \vartheta_q^s(t_k)\, y_s^k(t_k).$$

By taking the initial conditions into account, we get

$$y_i^k(t) = \sum_{s=1}^{n} A_{is}(t, t_k)\, R_s(t_k, [\varphi^k]),$$

with the notation

$$A_{is}(t, t_k) \equiv \sum_{q=1}^{n} \zeta_i^q(t)\, \vartheta_q^s(t_k).$$

By substituting the expression for $y_i^k(t)$ in (27) and introducing the additional notation $B_{is}(t_k) \equiv A_{is}(T, t_k)$, we obtain, for $t = T$

$$y_i([v]) \equiv y_i(T, [v]) = \sum_{k=1}^{j} \sum_{s=1}^{n} B_{is}(t_k)\, R_s(t_k, [\varphi^k])\ (i = 1, \ldots, n). \tag{28}$$

The matrix $B_{is}(t_k)$ is defined only by Eqs. (27) in the variations, and does not depend on the choice of the functions $\varphi^k(z)$. The vector $y([v]) \equiv (y_1, \ldots, y_n)$ is linearly expressed in terms of the vectors $R^k \equiv (R_1^k, \ldots, R_n^k)$ $(k = 1, \ldots, j)$.

The points of phase space of the form $x^1 + y([v])$, where $v \in \Gamma$, form a certain set Π. We now prove that Π is convex. Let $y([v'])$ and $y([v''])$ be arbitrary variations of the trajectories obtained with the modified controls v' and v'' with defining points and functions t'_α, $\varphi^\alpha(z)$ $(\alpha = 1, \ldots, j')$; t''_β, $\varphi^{*\beta}(z)$ $(\beta = 1, \ldots, j'')$ respectively. By definition

$$\pi' \equiv x^1 + y([v']) \in \Pi, \quad \pi'' \equiv x^1 + y([v'']) \in \Pi.$$

We show that any point of the form

$$\pi \equiv \mu\pi' + (1-\mu)\pi'' = x^1 + \mu y([v']) + (1-\mu)\, y([v'']), \quad 0 \leqslant \mu \leqslant 1, \tag{29}$$

which lies on the line segment joining π and π'' also lies in Π. For this, obviously, it is necessary to show that there exists a modified control $v \in \Gamma$ such that

$$y([v]) = \mu y([v']) + (1-\mu)\, y'([v'']). \tag{30}$$

We define control $\underline{v}$ in the following manner. As the control's defining points we take the union of the points t'_α and t''_β. The number of defining points of control $\underline{v}$ thus equals $j' + j'' - h$, where $\underline{h}$ is the number of pairs of coincident points t'_α and t''_β. The defining points of control $\underline{v}$ are divided into three classes: 1) the

points which coincide with the defining points of control v' and do not coincide with the defining points of control v"; 2) the points which coincide with defining points of control v" and do not coincide with defining points of control v'; 3) the points which coincide simultaneously with defining points of controls v' and v". For the points of the first class, $t_k = t'_{\alpha 1}$ (k and α_1 run through j'−h values), we give the defining functions of control $\underline{v}$ in the form

$$\varphi^k(z) = \begin{cases} \varphi'^{\alpha_1}\left(\frac{z}{\mu}\right) & 0 \leqslant z \leqslant \mu \\ u(t'_{\alpha_1}) & \mu < z \leqslant 1. \end{cases}$$

For the points of the second class $t_l = t''_{\beta 1}$ (l and β_1 run through j" −h values)

$$\varphi^l(z) = \begin{cases} \varphi'^{\beta_1}\left(\frac{z}{1-\mu}\right) & 0 \leqslant z \leqslant 1-\mu \\ u(t''_{\beta_1}) & 1-\mu < z \leqslant 1. \end{cases}$$

For points of the third class, $t_m = t'_{\alpha 2} = t''_{\beta 2}$ (m, α_2, β_2 run through $\underline{j}$ values)

$$\varphi^m(z) = \begin{cases} \varphi'^{\alpha_2}\left(\frac{z}{\mu}\right) & 0 \leqslant z \leqslant \mu \\ \varphi''^{\beta_2}\left(\frac{z-\mu}{1-\mu}\right) & \mu < z \leqslant 1. \end{cases}$$

By computing $R_i(t_k, [\varphi^k])$, $R_i(t_l, [\varphi^l])$ and $R_i(t_m, [\varphi^m])$ by formula (23), we find that

$$R_i(t_k, [\varphi^k]) = \mu R_i(t'_{\alpha_1}, [\varphi'^{\alpha_1}]),$$

$$R_i(t_l, [\varphi^l]) = (1-\mu) R_i(t''_{\beta_1}, [\varphi''^{\beta_1}]),$$

$$R_i(t_m, [\varphi^m]) = \mu R_i(t'_{\alpha_2}, [\varphi'^{\alpha_2}]) + (1-\mu) R_i(t''_{\beta_2}, [\varphi''^{\beta_2}]).$$

If we now determine $y_i([v])$ by (28), we convince ourselves of the validity of relationship (30). We also note that $x^1 \in \Pi$ since, by giving the modified control $v(t; \tau, t_1, \ldots, t_k; \varphi^1, \ldots, \varphi^k)$ such that; at each defining point t_k, the defining function $\varphi^k(z) \equiv u(t_k)$, we obtain $y_i([v]) = 0$.

Thus, Π is a convex set which includes point x^1. For definiteness, we consider further the case of a minimum of the functional S. The sets G^- and Π have the common point x^1. We now show that, in the nondegenerate problem, none of the points $g^- \in G^-$ (in particular, the point x^1) can be an interior point of Π. Indeed, let, on the contrary, point g^- be interior to Π. If the problem is nondegenerate, the set G^- will have interior points. Otherwise, hyperplane C would be a bracket for set G, and for any point $x = (x_1, \ldots, x_n) \in G$, the inequality

$\sum_1^n c_i(x_i - x_i^1) \geqslant 0$, would hold, this being the condition for a degenerate problem. Therefore, without any limitation on generality, we can assume that g^- is an interior point of G^- since, if g^- were a boundary point of G^-, one could always find another point interior for both G^- and Π (by virtue of the fact that, in any arbitrarily small neighborhood of any point in G^- there are contained points which are interior for G^-). We then find some neighborhood E of point g^-, all the points of which are interior in both Π and G^-. We choose the modified control $v \in \Gamma$ such that $x^1 + y([v]) = g^-$ (this is always possible, since $g^- \in \Pi$). The end of the trajectory $x(t, [v])$ is the point $x^1 + \delta x(T, [v])$, where $\delta x(T, [v]) = \tau y([v]) + \tau\epsilon(\tau)$. Since $\lim_{\tau \to 0} \epsilon(\tau) = 0$, we can always choose a τ sufficiently small so that the point $g \equiv x^1 + y([v]) + \epsilon(\tau) = g^- + \epsilon(\tau)$ lies in E. But points of the form $(1-\mu)x^1 + \mu g$, $0 < \mu \leq 1$, $g \in E$, are interior points of G^-. By choosing $\mu = \tau$, we are led to the conclusion that point $x^1 + \delta x(T, [v])$ is also an interior point of G^-, i.e., in other words, control $v(t; \tau, t_1, \ldots, t_k; \varphi^1, \ldots, \varphi^k)$ takes the system to the interior of set G^-. This contradicts the optimality of control u(t) and point $g^- \in G^-$ cannot be an interior point of Π. We show analogously that point $\pi \in \Pi$ cannot be interior of G^-.

It thus follows that the convex sets Π and G^- can have in common only boundary points, one of which (or the only one of which) is the point x^1, where one of the sets G^- has interior points in phase space X. Therefore,

hyperplane A, defined by the expression $\sum_1^n a_s (x_s - x_s^1) = 0$, may be passed through point x^1, thus separating sets G^- and Π. The signs of coefficients a_s can be so chosen that the set Π is located on that side of hyperplane A where $\sum_1^n a_s (x_s - x_s^1) \geqslant 0$. (set G^- is then completely located on the side of the hyperplane where $\sum_1^n a_s (x_s - x_s^1) \leqslant \leqslant 0$). Therefore, for any control $v(t; \tau, t_1, \ldots, t_k; \varphi^1, \ldots \varphi^k) \in \Gamma$, the inequality $\sum_1^n a_s y_s ([v]) \geqslant 0$, will hold, i.e.

$$\lim_{\tau \to 0} \frac{\sum_1^n a_s \delta x_s (T, [v])}{\tau} \geqslant 0.$$

We now consider the functional $\bar{S} = \sum_1^n a_s x_s (T)$. Then, the requirement that $\lim_{\tau \to 0} \frac{\sum_1^n a_s \delta x_s}{\tau} \geqslant 0$ means that

$$\lim_{\tau \to 0} \frac{\delta \bar{S}}{\tau} \geqslant 0. \tag{31}$$

As a modified control, we select the control $v \in \Gamma$ with only one defining point t_1, such that $\delta u(t) \equiv v(t) - u(t)$ may differ from zero only on the segment $[t_1 - \tau, t_1]$. For the control $v(t) = u(t) + \delta u(t)$, inequality (31) holds (as for all controls in Γ). By now introducing the vector p(t) and the function $H(x, p, u, t) \equiv \sum_1^n p_i f_i$ $(x_1, \ldots, x_n; u_1, \ldots, u_r; t)$, which satisfy the relationships

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, \ldots, n),$$

and by setting $p_i(T) = -a_i$ $(i = 1, \ldots, n)$, we can use, for the functional $\bar{S} = \sum_1^n a_s x_s (T)$, the formula for the increment of functional values (Appendix to Part I). By repeating virtually word for word the argument in the proof of Theorem 1, we can convince ourselves that condition (31) can hold only in the case, when at any moment of time t, the function $K(t,u) \equiv H[x(t), p(t), u, t]$ assumes its maximum value with respect to u.

The case of the maximum of functional $S = \sum_1^n c_i x_i (T)$ is treated analogously. Thus, Theorem 3 is proved.

2°. The Proof of Theorem 4

We carry out the proof for the case when one seeks a minimum of the functional S. The case, where the maximum of the functional is sought, is treated analogously.

Let W_T be the set of points in phase space X which are attainable from point x^0 at time $t = T$ for all the possible controls $u(t) \in U$. We consider the intersection Q of sets G and W_T: $Q = G \cap W_T$. The control u(t) and the trajectory x(t), corresponding to it, are min-optimal if, for each point $x(x_1, \ldots, x_n) \in Q$, the inequality $\sum_1^n c_i x_i (T) \leqslant \sum_1^n c_i x_i$ holds. By denoting $x_i(T) = x_i^1$, we get

$$\sum_1^n c_i (x_i - x_i^1) \geqslant 0. \tag{32}$$

To prove Theorem t, it suffices to show that, if the conditions of the Theorem hold, then inequality (32) is satisfied for each point $x \in Q$.

We use the notation

$$a_i = \lambda c_i + \mu b_i \quad (i = 1, \ldots, n). \tag{33}$$

If the control u(t) satisfies the maximum condition with respect to the vector p(t) which, at time t = T, assumes the value

$$p(T) = -a, \quad a = (a_1, \ldots, a_n),$$

then, in accordance with Theorem 2 of the present work, control u(t) is min-optimal with respect to $S = \sum_1^n a_i x_i(T)$ in the problem with free trajectory right ends, i.e., for each point $x \in W_T$, the following inequality is valid

$$\sum_1^n a_i (x_i - x_i^1) \geqslant 0. \tag{34}$$

On the other hand, for each point $x \in G$ the following inequality holds, by definition,

$$\sum_1^n b_i (x_1^1, \ldots, x_n^1)(x_i - x_i^1) \geqslant 0. \tag{35}$$

Therefore, for each point $x \in Q = G \cap W_T$, inequalities (34) and (35) hold.

But it follows from (33) that, for $\lambda > 0$,

$$c_i = \frac{1}{\lambda} a_i - \frac{\mu}{\lambda} b_i \qquad (i = 1, \ldots, n).$$

We now choose an arbitrary point $x \in Q$ and set up the expression

$$\sum_1^n c_i (x_i - x_i^1) = \frac{1}{\lambda} \sum_1^n a_i (x_i - x_i^1) - \frac{\mu}{\lambda} \sum_1^n b_i (x_i - x_i^1).$$

By virtue of inequalities (34) and (35), inequality (32) follows from this. g.e.d.

3°. The Proof of Theorem 5

For definiteness, we consider the case when the control satisfies the maximum condition. The case of the minimum condition is handled analogously.

We introduce the difference

$$\Delta M \equiv H[y(t + \Delta t), u(t + \Delta t), t + \Delta t] - H[y(t), u(t), t].$$

Now, to shorten the formulas, we introduce the vector $y = (y_1, \ldots, y_{2n})$ ($y_1 = X_1$; $y_{n+1} = p_i$; $i = 1; \ldots, n$). By virtue of the maximum condition

$$H[y(t + \Delta t), u(t + \Delta t), t + \Delta t] \geqslant H[y(t + \Delta t), u(t), t + \Delta t],$$
$$H[y(t), u(t), t] \geqslant H[y(t), u(t + \Delta t), t].$$

From whence follows, for ΔM,

$$H[y(t + \Delta t), u(t), t + \Delta t] - H[y(t), u(t), t] \leqslant \Delta M \leqslant$$
$$\leqslant H[y(t + \Delta t), u(t + \Delta t), t + \Delta t] - H[y(t), u(t + \Delta t), t]. \tag{36}$$

Because of the continuity of $y_s(t)$ ($s = 1, \ldots, 2n$) in time, and of the continuity of $H(y, u, t)$ over the set of arguments, for $\Delta t \to 0$, the extreme terms in inequality (36) have the limit zero. Consequently, $\lim \Delta M = 0$, which proves the continuity of the function $M(t)$.

Now, let $\underline{t}$ be a continuity point of control $u(t)$. Then, $\lim_{\Delta t \to 0} u(t + \Delta t) = u(t)$ and, moreover, the derivatives $\dot{y}_s(t)$ ($s = 1, \ldots, 2n$) exist and are continuous. By dividing (36) by Δt and letting Δt approach zero, we convince ourselves that the limits for the extreme terms of the inequality exist and equal

$$\sum_{1}^{2n} \frac{\partial H[y(t), u(t), t]}{\partial y_s} \dot{y}_s(t) + \frac{\partial H[y(t), u(t), t]}{\partial t}.$$

By virtue of the relationships, $\dot{y}_i = \partial H / \partial y_{n+1}$, $\dot{y}_{n+1} = -\partial H / \partial y_i$ ($i = 1, \ldots, n$), the sum in the previous expression reduces to zero and the limit of both extreme terms equals $\partial H / \partial t$, thanks to which, $dM/dt = \partial H / \partial t$. Consequently, the theorem is proven.

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ON THE SYMMETRIC PERIODIC MOTIONS OF MULTISTAGE RELAY SYSTEMS

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Many works [1-15] have been devoted to the exposition of various methods of seeking, and investigating the stability of the simplest symmetric-periodic motions of linear systems. Systems with two relays [13, 16] and, in particular, a two-stage relay system [17], were created with these methods. However, the complexity in the use of these methods increases with an increasing number of stages in the relay system under consideration.

In the present work, we solve the problem of finding, and investigating the stability of, the simplest symmetric motions of multistage relay systems. The solution is attained by the use of the so-called phase characteristic [15] which is of independent interest.

1. The Phase Characteristic

Let a relay with the symmetric characteristic, represented in Fig. 1 be connected to the output of a linear link. If an infinite series of symmetric rectangular pulses with half-period T is applied to the input of a linear link of this system then, at the relay output, there is produced some series of rectangular pulses. For some intervals of values of the half-period T, the series of output pulses will also by symmetric and periodic, with period 2T, but with the moments of switching shifted with respect to the moments of switching of the input pulses by the quantity $\theta(T)$. The curve $\theta(T)$ will be called the phase characteristic of the open-loop relay system under consideration by us.

As is well known, the output of a linear link may be expressed in terms of its transfer conductance function $\varphi(t)$ in the form [12-13]

$$y(t, T) = \sum_{k=0}^{\infty} (-1)^{k-1} [\varphi(t - (k-1)T) - \varphi(t - kT)] + \varphi(t). \tag{1}$$

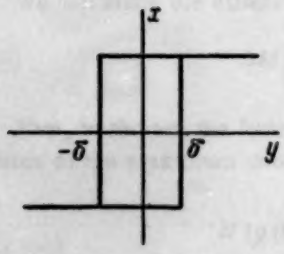

Fig. 1

The moment $\theta(T)$ of relay switching from -1 to 1 which is closest to time $t = 0$ is found as the least positive root of the equation

$$y(t, T) = \delta, \tag{2}$$

for which the conditions of switching hold. In the case when y_t' is continuous, the switching condition can be written in the form

$$y_t'(t, T) > 0.$$

In seeking the phase characteristic $\theta(T)$ in practice, one should use formula (1) if the series in (1) converges rapidly and the necessary accuracy can be attained after not too many of its terms have been summed. If the transfer constant K(p) of the linear link can be represented to a good enough approximation in the form

$$K(p) = \sum \frac{c_j}{p - \lambda_j}, \tag{3}$$

then, as is well known [12], the linear link's output can be written in finite form:

$$y(t, T) = -\sum \frac{c_j}{\lambda_j} e^{\lambda_j t} \operatorname{th} \frac{\lambda_j T}{2} + \sum \frac{c_j}{\lambda_j} (e^{\lambda_j t} - 1). \tag{4}$$

Finally, it is well known [14] that the sum in (1) can be transformed to the form

$$y(t, T) = -\frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\left| K\left(i(2m-1)\frac{\pi}{T}\right)\right|}{2m-1} \times \tag{5}$$
$$\times \sin\left[(2m-1)\frac{\pi}{T} t + \arg K\left(i\frac{(2m-1)\pi}{T}\right)\right],$$

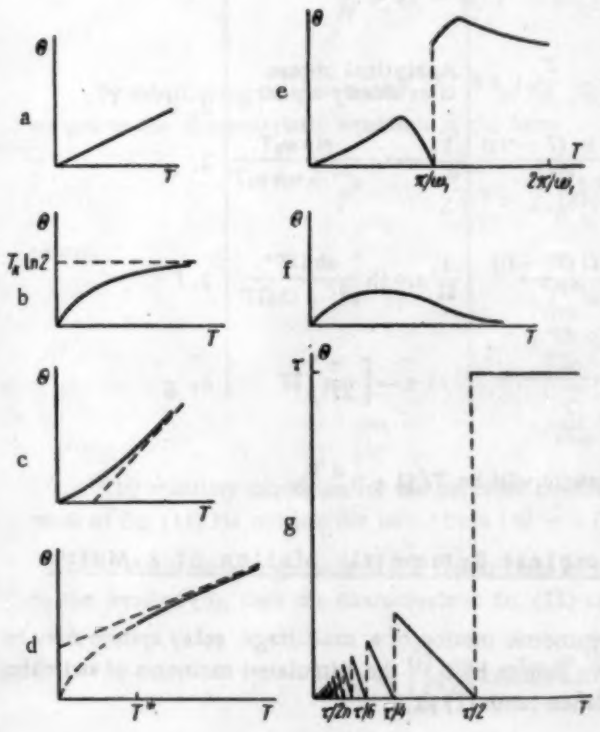

Fig. 2

which it is natural to use if series (5) converges sufficiently rapidly, i.e., if the quantities $K\left(1\frac{\pi}{T}\right)$, $K\left(1\frac{3\pi}{T}\right)$, . . . tend to zero with sufficient rapidity.

We give the easily found phase characteristic of standard linear links of automatic control systems (Fig. 2) with relays for which $\delta = 0$.

We note that when a lag τ appears in the linear link, the phase characteristic $\theta(T)$ is replaced by $\theta(T) + \tau - [\tau/2T]2T$, where $[\tau/2T]$ denotes the greatest integer less than $\tau/2T$.

We now consider the open chain consisting of m series-connected linear links and relays. Let $\theta_1(T)$ be the phase characteristc of the first linear element and the relay following it, $\theta_2(T)$ the phase characteristic of the second linear element and the relay following it, etc. Obviously, to within a multiple of the period, the phase characteristic $\theta(T)$ equals $\theta_1(T) + \theta_2(T) + \ldots + \theta_m(T)$ or, if we understand by $\theta(T)$ the moment of relay switching which is shifted in time by less than 2T:

$$\theta(T) = \theta_1(T) + \ldots + \theta_m(T) - \left[\frac{\theta_1(T) + \ldots + \theta_m(T)}{2T}\right] 2T, \tag{6}$$

where [ ] denotes the integral part of its argument.

2. Seeking the Simplest Symmetric-Periodic Motions of a Multistage Relay System

We consider a multistage relay system consisting of m series-connected linear links and m symmetric relays. The phase characteristic $\theta(T)$ corresponding to the open-loop system can be found in terms of phase characteristics of the constituent single-relay systems in the form of (6). Obviously, the closed-loop relay system will admit the simplest symmetric motion with half-period T if only

$$\theta(T) = 0, \tag{7}$$

and, consequently, the search for the simplest symmetric motions reduces to finding the points of intersection of the curve $\theta^*(T) = \theta_1(T) + \ldots + \theta_m(T)$ with the lines $\theta^*(T) = 2kT$, where $k = 0, 1, 2 \ldots$

№	Transmission factor of the linear link	Output equation of the linear link in the interval $0 < t < T$	Phase characteristic $\theta(t)$	Figure No.
1	$\frac{1}{p}$	$-\frac{T}{2}+t$	$\frac{T}{2}$	2, a
2	$\frac{1}{T_k p+1}$	$-e^{-\frac{t}{T}}\operatorname{th}\frac{T}{2T_k}-e^{\frac{t}{T_k}}+1$	$T_k \ln\frac{2}{1+e^{-\frac{T}{T_k}}}$	2, b
3	$\frac{1}{T_k p-1}$	$-e^{-\frac{t}{T_k}}\operatorname{th}\frac{T}{2T_k}+e^{\frac{t}{T_k}}-1$	$T_k \ln\frac{1+e^{\frac{T}{T_k}}}{2}$	2, c
4	$\frac{1}{p^2}$	$-\frac{tT}{2}+\frac{t^2}{2}$	T	
5	$\frac{1}{p(p+\lambda)}$	$-\frac{1}{\lambda}\left(\operatorname{th}\frac{\lambda T}{2}+1\right)e^{-\lambda t}+\frac{1}{\lambda}-\frac{T}{2}+t$	Analytical phrase is evidently absent	2, d
6	$\frac{p}{p^2+2\delta p+\omega_0^2}$	$\frac{2e^{\delta(T-t)}\left(e^{\delta T}\sin\omega_1 t-\sin\omega_1(T-t)\right)}{\omega_1\left(1+2e^{\delta T}\cos\omega_1 T+e^{2\delta T}\right)}$ For $\omega_1^2=\omega_0^2-\delta^2>0$	$\frac{1}{\omega_1}\operatorname{arctg}\frac{\sin\omega_1 T}{e^{\delta T}+\cos\omega_1 T}$	2, e
		$\frac{2e^{\delta(T-t)}\left(e^{\delta T}\operatorname{sh}\Omega t-\sin\Omega(T-t)\right)}{\Omega\left(1+2e^{\delta T}\operatorname{ch}\Omega T+e^{2\delta T}\right)}$ For $\Omega^2=\delta^2-\omega_0^2\geqslant 0$	$\frac{1}{\Omega}\operatorname{arc\,th}\frac{\operatorname{sh}\Omega T^*}{e^{\delta T}+\operatorname{ch}\Omega T}$	2, f
7	$e^{-p\tau}$	—	$\tau-\left[\frac{\tau}{2T}\right]2T$	2, g

For $\Omega = 0$, the equation of the phase characteristic will be $T/(1 + e^{\delta T})$.

8. Investigation of the Stability of the Simplest Symmetric Motion of a Multistage Relay System

The investigation of the stability of the simplest symmetric motion of a multistage relay system may be easily carried out by the method described in [12, 13]. We denote by $t_k^{(i)}$ the stimulated moments of switching of relay i, and we write the linear link's output in accordance with (1) as

$$y_i(t) = \sum_{k=n}^{-\infty}(-1)^{k-1}\left[\varphi_i\left(t-t_{k-1}^{(i)}\right)-\varphi_i\left(t-t_k^{(i)}\right)\right]+(-1)^{n-1}\varphi_i\left(t-t_n^{(i)}\right), \tag{8}$$

where φ_i is the transfer conductance function of the i'th linear link and n is the ordinal number of last relay switching up to time t. Let $nT + \alpha_1, \ldots, nT + \alpha_m$ be the switching moments of, respectively, relays 1, 2, ..., m which lie in the interval $[t-T, t]$. The following relay switching in the interval $(t, t+T]$ will occur in correspondence with the order of increasing α_i. Therefore, the order of relay switching may be presented in the form of the substitution $\begin{pmatrix}1 & 2 & \ldots & m\\ r & r_2 & \ldots & r_m\end{pmatrix}$. By virtue of this, linearized Eq. (8) is written in the following form:

$$dt_{n+1}^{(i+1)}\left\{\sum_{k=n+\varepsilon_i}^{-\infty}(-1)^{k-1}\left[\varphi_i'\left((n+1)T-(k-1)T+\alpha_{i+1}-\alpha_i\right)-\right.\right.$$
$$\left.\left.-\varphi_i'\left((n+1)T-kT+\alpha_{i+1}-\alpha_i\right)\right]+\right.$$

$$+(-1)^{n+\varepsilon_i-1}\varphi_i'((1-\varepsilon_i)T+\alpha_{i+1}-\alpha_i)\Big\}+$$
$$+2(-1)^{n+\varepsilon_i-2}\varphi_i'((2-\varepsilon_i)T+\alpha_{i+1}-\alpha_i)\,dt^{(i)}_{n+\varepsilon_i}+ \tag{9}$$
$$+2(-1)^{n+\varepsilon_i-3}\varphi_i'((3-\varepsilon_i)T+\alpha_{i+1}-\alpha_i)\,dt^{(i)}_{n+\varepsilon_i-1}+\ldots=0$$
$$(i=1,2,\ldots,m),$$

where ϵ_i equal zero or one depending on whether $r_i > r_{i+1}$ or $r_i < r_{i+1}$. Equation (9) allows one to determine the variations $dt^{(i+1)}_{n+1}$ of the moments of switching of the $(i+1)$'st relay as functions of the stimuli $dt^{(i)}_{n+\epsilon_i}$, $dt^{(i)}_{n-1+\epsilon_i}$, ... of the i'th relay. From the conditions for relay switching in the steady state, the following relationships are easily established:

$$\gamma_i=(1-\varepsilon_i)T+\alpha_{i+1}-\alpha_i=\begin{cases}\theta_i & \text{for } \theta_i\leqslant T\\ \theta_i-T & \text{for } \theta_i>T\\ T & \text{for } \theta_i=0.\end{cases} \tag{10}$$

By substituting $dt^{(i)}_k = \alpha^{(i)}z^{n+i-k}$ in Eq. (9) and then eliminating the unknown $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$, we get to the characteristic equation in the form

$$\chi(z)=\chi_1(z),\ldots,\chi_m(z)-1=0, \tag{11}$$

where

$$\chi_i(z)=\frac{z\sum\limits_{k=0}^{+\infty}(-1)^{k-1}\varphi_i'(kT+\gamma_i)\,z^k}{\sum\limits_{k=0}^{\infty}(-1)^{k-1}\varphi_i'(kT+\gamma_i)}.$$

The stability condition for the periodic motion under consideration consists of this, that all but one of the roots of Eq. (11) lie outside the unit circle $|z|\leq 1$ [12, 13].

If the transfer constants of the linear links of the multistage relay system under consideration are written in the form of (3), then the characteristic Eq. (11) can be written in the explicit form:

$$\chi(z)=\prod_{i=1}^{m}\sum\frac{c_{ij}\,ze^{\lambda_{ij}\gamma_i}}{1+ze^{\lambda_{ij}T}}\left(\sum\frac{c_{ij}\,e^{\lambda_{ij}\gamma_i}}{1+e^{\lambda_{ij}T}}\right)^{-1}-1=0. \tag{12}$$

In case the linear links contain lags τ_i, the functions $\chi_i(z)$ entering into characteristic Eq. (11) will have the following form:

$$\chi_i(z)=\sum\frac{c_{ij}\,z^{s_i+1}e^{\lambda_{ij}\beta_i}}{1+ze^{\lambda_{ij}T}}\left(\sum\frac{c_{ij}\,e^{\lambda_{ij}\beta_i}}{1+e^{\lambda_{ij}\beta_i}}\right)^{-1}, \tag{13}$$

where s_i is the least positive integer for which $s_iT+\gamma_i-\tau_i>0$.

If the series entering into (11) converge sufficiently rapidly for $|z|=1$, then numerical investigation of stability may be carried out, for example, by means of the reductions described in [13].

In case the frequency characteristics $K_j(i\omega)$ of the linear links are given, the characteristic Eq. (11) by the use of the relationship [14]

$$\sum_{k=-\infty}^{\infty} \varphi_j'(2kT+t)\, e^{2ik\varphi} = \frac{e^{-\frac{i\varphi t}{T}}}{2T} \sum_{m=-\infty}^{\infty} e^{m\frac{\pi}{T}ti} K_j\left(m\frac{\pi}{T}i - \frac{\varphi}{T}i\right), \tag{14}$$

may be given in the following form:

$$\chi(e^{i\varphi}) = \chi_1(e^{i\varphi}) \ldots \chi_m(e^{i\varphi}) - 1 = 0, \tag{15}$$

where

$$\chi_j(e^{i\varphi}) = e^{i\left(\varphi - \frac{\gamma_j}{T}\varphi\right)} \sum_{m=-\infty}^{\infty} e^{i(2m-1)\frac{\pi}{T}} K_j\left(\frac{i(2m-1)\pi}{T} - \frac{\varphi}{T}\right) \times$$
$$\times \left[\sum_{m=-\infty}^{+\infty} e^{i(2m-1)\frac{\pi}{T}} K_j\left(\frac{(2m-1)\pi}{T}i\right)\right]^{-1}. \tag{16}$$

If series (16) converge sufficiently rapidly, then investigation of stability is conveniently carried out by the numerical construction of the boundaries of the stable regions which are obtained as φ varies from 0 to 2π.

4. Examples of the Construction of the Phase Characteristics, the Finding, and the Investigation of the Stability of the Periodic Motions

1. Let the graph of the transfer conductance of the linear link be given. In this case, the phase characteristic can be constructed by the following method. For each fixed value of the half-period T we find the output of the linear link, for which we geometrically sum the reactions of the linear link to the rectangular pulses in accordance with formula (1) and we determine the moment of switching of the relay placed after the linear link*. By constructing the outputs of the linear links and determining the shifts of the moments of switching of the relays for various values of T, we can find the graph of the phase characteristic. Thus, for the transfer conductance function, represented on Fig. 3,a, the phase characteristic found by this method has the form shown on Fig. 3,b.

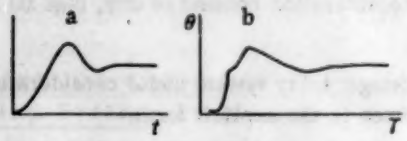

Fig. 3

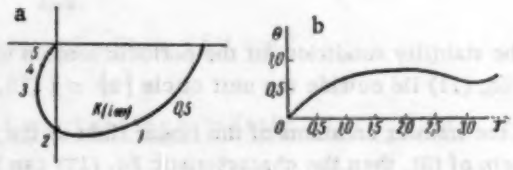

Fig. 4

2. If the frequency characteristic $K(i\omega)$ of the linear link is given then, to find the phase characteristic, it is convenient to use formula (5). In this case, the construction of the linear link's output reduces to the geometric summing of the reactions of the linear link to the harmonic frequencies π/t, $3\pi/T$, $5\pi/T$, . . ., the amplitudes and phases of which are directly determined from the frequency characteristics as the modulus and argument of the quantities $K(i\pi/T)$, $K(3\pi i/T)$. By determining the shift θ for different values of T, we can construct the graph of the phase characteristic. Thus, for the frequency characteristic shown in Fig. 4,a, the phase characteristic has the form shown in Fig. 4,b.

3. To illustrate the theory presented in this paper, we consider the m-stage relay system whose linear links have one of the following transfer constants, $e^{-p\tau}/(T_k p + 1)$, $e^{-p\tau}/(T_k p - 1)$, $e^{-p\tau}/p$. In accordance with Section 2 of this paper, the half-periods of the possible periodic motions are found from the condition that the phase characteristic of the open-loop relay system intersect the lines $\theta = 2kT$, $k = 1, 2, \ldots$. Thus, for the four-stage relay system whose linear links are identical and have the transfer constant $e^{-p\tau}/(T_k p + 1)$, the half-periods of the symmetric periodic motions are found from the condition that the phase characteristic of the linear link intersect the lines $\theta^* = kT/2$, $k = 1, 2, 3, 4$ (Fig. 5).

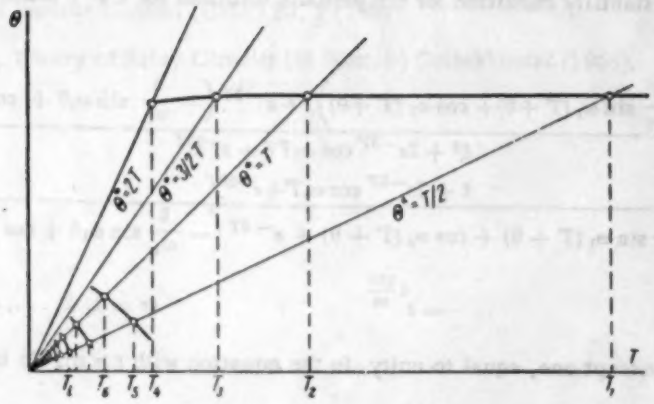

Fig. 5

If the number of stages is greater than two, all the symmetric motions will be unstable. Indeed, the characteristic equation for such a system, after some simple transformations, can be written in the following form:

$$(\xi^{m+N} - 1) + a_1(\xi^{N+m-1} - 1) + \ldots + a_{N+m}(\xi - 1) = 0, \tag{17}$$

where $a_i > 0$. Dividing the left member of Eq. (17) by $\xi - 1$, we find that

$$\xi^{N+m-1} + \ldots + b_{N+m-2}\xi + 1 + B = 0, \tag{18}$$

where $B > 0$. Since the product of the roots of Eq. (18) is equal to $1 + B$, there are roots which do not lie within the unit circle, and, consequently, the symmetric periodic motion is unstable. By virtue of this, in the multistage relay systems considered here for which $m \geq 2$, only more complex periodic motions are possible.

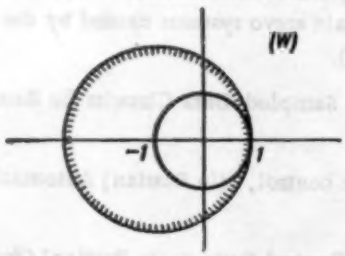

Fig. 6

We now consider an m-stage system whose linear links are also identical and are described by the transfer constants $\frac{p}{p^2 + 2\delta + \omega_0^2}$, $\omega_0^2 - \delta^2 = \omega_0^2 > 0$. In this case, the half-periods of the possible periodic motions may be determined from the conditions that the phase characteristic of the linear link, the form of which is shown in Fig. 2,e, intersect the lines $\theta^* = 2kT/m$, $k = 0, 1, \ldots$. Thus, for $k = 0$, it is clear that the periodic mode with half-period π/ω_1 is possible in the system.

The question of the stability of the periodic modes in a system with identical linear links leads to the investigation of the roots of the equation

$$(\chi(z))^m - 1 = 0. \tag{19}$$

In our case, each individual factor of this equation for the periodic mode with half-period π/ω_1 is written in the form

$$\frac{\xi - e^{-\delta\frac{\pi}{\omega_1}}}{1 - e^{-\delta\frac{\pi}{\omega_1}}} = W, \tag{20}$$

where $z = 1/\xi$ and W takes the values $e^{-12\pi r/m}$, $r = 1, 2, \ldots, m$.

By substituting $\xi = e^{i\varphi}$ in (20) and constructing the graph of the regions of stability for the parameter W, we find that for any number of links there will be stability, since the unit circle lies completely inside the region

of stability (Fig. 6). The stability condition for the periodic solutions for $k \neq 0$ leads to the requirement that the roots of all the equations

$$\frac{\xi\left(-\frac{\delta}{\omega_1}\sin\omega_1(T+\theta)+\cos\omega_1(T+\theta)\right)+e^{-\delta T}\left(-\frac{\delta}{\omega_1}\sin\omega_1\theta+\cos\omega_1\theta\right)}{\xi^2+2e^{-\delta T}\cos\omega_1 T\,\xi+e^{-2\delta T}}\times$$

$$\times\frac{1+2e^{-\delta T}\cos\omega_1 T+e^{-2\delta T}}{-\frac{\delta}{\omega_1}\sin\omega_1(T+\theta)+\cos\omega_1(T+\theta)+e^{-\delta T}\left(-\frac{\delta}{\omega_1}\sin\omega_1\theta+\cos\omega_1\theta\right)}= \tag{21}$$

$$=e^{i\frac{2\pi r}{m}}\qquad(r=1,\ 2,\ \ldots,\ m)$$

of the second order in ξ, except one, equal to unity, in the equation with $r = m$, lie inside the unit circle.

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APPROXIMATE DETERMINATION OF THE AUTO-OSCILLATION PARAMETERS IN RELAY SYSTEMS WITH LAGGED FEEDBACK

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An approximate method is presented for determining the parameters of the auto-oscillations of one class of relay systems on the basis of a modification of the harmonic balance method.

The harmonic balance method [1, 2] is the most widely used method of determining the auto-oscillation parameters in relay systems of automatic control. An exact method of determining the periodic modes was developed for relay systems [3, 4]. It turns out that an entire group of relay systems can not be subjected to investigation by the harmonic balance method [5], namely, systems with lagged feedback (LF). We now give an elementary example wherein the harmonic balance method does not give the correct answer to the question of auto-oscillations. For this, we consider the simplest saw-tooth generator, the block schematic of which is given in Fig. 1, a. It consists of a linear portion, in the form of an aperiodic link, and a relay element with hysteresis $\varkappa_0$. The solution by the harmonic balance method does not give an intersection of the frequency characteristic $W(j\omega)$ with the curve of the nonlinear impedance $1/I(A)$ (Fig. 1, b), i.e., it shows the absence of auto-oscillations.

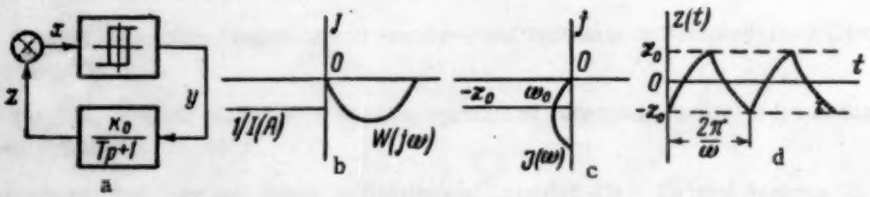

Fig. 1

The solution by the frequency (exact) method [4] gives the exact value of the frequency and "amplitude" of the auto-oscillation (Fig. 1, c). In the given case

$$\operatorname{Im} J(\omega) = -z\left(\frac{\pi}{\omega}\right) = -k_0 \operatorname{th} \frac{\pi}{2\omega T} = -\varkappa_0, \tag{1}$$

whence

$$\omega_0 = \frac{\pi}{T \ln \frac{k_0 + \varkappa_0}{k_0 - \varkappa_0}}. \tag{2}$$

The maximum deviation z(t) from the equilibrium position is

$$[z(t)]_{\max} = \varkappa_0, \tag{3}$$

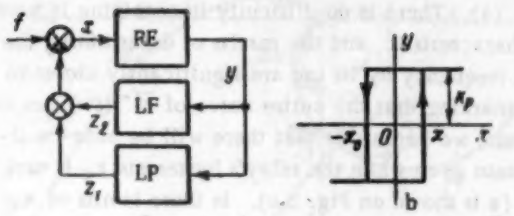

Fig. 2

i.e., the auto-oscillation occurs inside the relay's hysteresis loop (Fig. 1,d).

By a relay system with LF we mean a system with an arbitrary linear portion (LP) (Fig. 2, a) in which the relay element (RE) is shunted by an internal feedback path (LF) in the form of an aperiodic or integrating link [5]. One can sometimes reduce to such a system a relay system in which the derivative of the time characteristic $\dot{h}(0) > 0$ (for $h(0) = 0$) or, what amounts to the same thing, the difference in the degrees of the denominator and the numerator of the linear portion's transfer function W(p) equals one. For the concrete examples in [5] there were given the qualitative and quantitative differences of the results of computing the auto-oscillations in a system with LF by the exact method of Ya. Z. Tsypkin [4] and the approximate method of L. S. Gol'dfarb [1]. The relays were chosen without dead zones (Fig. 2,b). For convenience in comparing the results of this paper and those of [5], we consider the same numerical example of computing the auto-oscillation parameters in a relay system with LF:

$$W_1(p) = \frac{k_1}{p(T_1p+1)(T_2p+1)} = \frac{75}{p(0.05p+1)(0.03p+1)},$$
$$W_2(p) = \frac{k_0}{T_0p+1} = \frac{1}{0.01p+1}, \tag{4}$$

where $W_1(p)$ is the transfer function of the linear portion of the system without feedback and $W_2(p)$ is the feedback's transfer function.

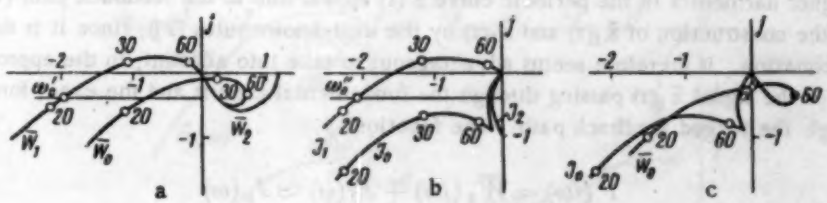

Fig. 3

On Fig. 3,a there are constructed the frequency characteristic $\bar{W}_0(j\omega) = \bar{W}_1(j\omega) + \bar{W}_2(j\omega) = \frac{4k}{\pi} W_0(j\omega)$ and the reciprocal of the relay's nonlinear gain I(A).

Figure 3,b gives the characteristic $J_0(\omega)$ of the relay system as the sum of the characteristics of the inherent system $J_1(\omega)$ and of the internal feedback path $J_2(\omega)$ (the "left" characteristic).

Figure 3,c gives, in the same scale, the characteristics $\bar{W}_0(j\omega)$ and $J_0(\omega) = J_1(\omega) + J_2(\omega)$. As we see, these characteristics are very remote from each other.

By using the well-known formula [5]

$$J(\omega) = k_0\left\{-\frac{\dot{h}(0)}{\omega} + \frac{4}{\pi}\sum_{m=1}^{\infty} U[(2m-1)\omega] + j\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{V[(2m-1)\omega]}{2m-1}\right\} \tag{5}$$

and by limiting ourselves to the first harmonic, we can write

$$J(\omega) \approx -k_p\frac{\dot{h}(0)}{\omega} + \frac{4k_p}{\pi}W(j\omega) = \bar{W}(j\omega) - k_p\frac{\dot{h}(0)}{\omega} = \bar{W}^{(-)}(j\omega). \tag{6}$$

It follows from this that a first approximation (in the sense of a first harmonic) to $J(\omega)$ differs from the given characteristic $\bar{W}(j\omega)$ by the quantity $k_p\dot{h}(0)/\omega$, completely defined by the ordinary parameters of the lagged feedback, since $\dot{h}(0) = k_0/T_0$ [5].

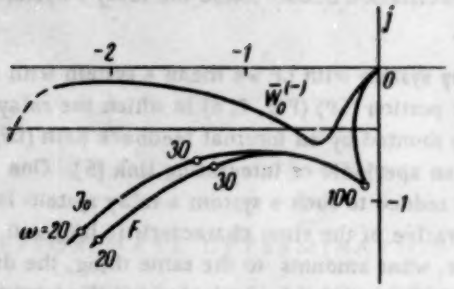

Fig. 4

Figure 4 gives the characteristic $\bar{W}^{(-)}(j\omega)$ of the system described by Eq. (4). There is no difficulty in obtaining it from the frequency characteristic, and the results of determining the auto-oscillation frequency by its use are significantly closer to the truth. By remarking that the entire curve of $\bar{W}^{(-)}(j\omega)$ lies in the left half-plane, we can assert that there will be auto-oscillation in the system even when the relay's hysteresis κ_0 is such that $a > \kappa_0 > 0$ (a is shown on Fig. 3,c). In these limits of κ_0, the harmonic balance method shows an absence of auto-oscillation (since the semiline -1 $K(A) = -\kappa_0$ lies completely in the left half-plane). However, in those cases when the Gol'dfarb method does give some result in determining the auto-oscillation frequency, this result is not changed by replacing $\bar{W}(j\omega)$ by the characteristic $\bar{W}^{(-)}(j\omega)$, since the auto-oscillation frequency is defined by the characteristics' imaginary components, and these are identical (6).

The quantitative difference in the solution by the exact and the approximate methods, as before, remain essential, and this is explained by the sharp difference of the form of the auto-oscillation from the sinusoidal, i.e., by the low filtering properties of systems with LF.

It was shown in [4] that the determination of the auto-oscillation frequency by the harmonic balance method in relay systems of the type in (4) without internal feedback ($k_0 = 0$) gives quite accurate results. This is also clear from Fig. 3. By determining the auto-oscillation frequencies in the points of intersection of the line $-\kappa_0$ with the curves $\bar{W}_1(j\omega)$ and $J_1(\omega)$ respectively, we obtain close values of ω_0' and ω_0''. This fact shows that the linear portion of system (4) itself is a good low-frequency filter. The inclusion of a lagged feedback changes the picture siginificantly. Higher harmonics in the periodic curve $\bar{z}(t)$ appear due to the feedback path (this assertion is easily verified by the construction of $\bar{z}_1(t)$ and $\bar{z}_2(t)$ by the well-known rules [2]), since it is described by a first-order differential equation. It therefore seems advantageous to take into account, in the approximate solution, the first harmonic of the signal $\bar{z}_1(t)$ passing through the fundamental circuit and the exact form of the signal $\bar{z}_2(t)$ passing through the lagged feedback path. The function

$$F(j\omega) = \bar{W}_1(j\omega) + J_2(\omega) \approx J_0(\omega) \tag{7}$$

will satisfy the problem posed. Its construction is much simpler than that of $J_0(\omega)$. For a lagged feedback in the form of an aperiodic link as in (4), the construction of the characteristic

$$J_2(\omega) = -k_0 k_p \left\{ \frac{1}{\omega T_0} \left(1 - \operatorname{th} \frac{\pi}{2\omega T_0} \right) + j \operatorname{th} \frac{\pi}{2\omega T_0} \right\} \tag{8}$$

poses no great difficulties. For a lagged feedback in the form of an integrating link, $W_2(p) = k_0/p$, the characteristic

$$J_2(\omega) = -\frac{k_p k_0}{\omega} \left(1 + j \frac{\pi}{2} \right) \tag{9}$$

is a straight line. When the transfer function of the system's linear portion, $W_1(p)$ is unknown, and only its frequency characteristic is available, $W_1(j\omega)$, the characteristic $F(j\omega)$ is determined by the direct graphical addition of $\bar{W}(j\omega)$ and $J_2(\omega)$.

The characteristic $F(j\omega)$ for system (4) was constructed on Fig. 4. As we see, it differs slightly from the exact characteristic $J_0(\omega)$, and the error in determining the auto-oscillation frequency is insignificant for any κ_0 or, what is equivalent, for any gain k_0 in the internal feedback path.

We also note that the harmonic balance method can give erroneous results even in computation for those relay systems in which $h(0) = \dot{h}(0) = 0$. As an example, we take the system with a transfer function $W_1(p)$ as given above in (4) and with an internal feedback transfer function

$$W_2(p) = \frac{k_0}{(T_0 p + 1)^2} = \frac{2{,}5}{(0.01p+1)^2}. \tag{10}$$

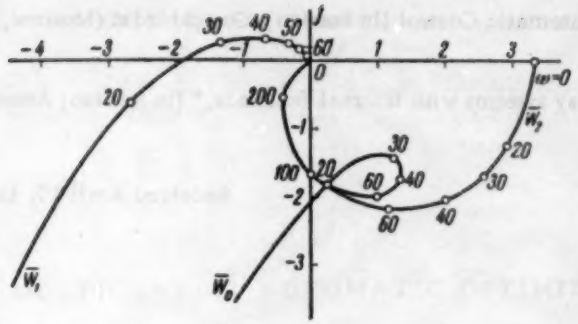

Fig. 5

Figure 5 shows the components, $\bar{W}_1(j\omega)$ and $\bar{W}_2(j\omega)$, and the total, $\bar{W}_0(j\omega)$ of the given system frequency characteristic. It is clear from the figure that, for certain values of ω, the characteristic $W_0(j\omega)$ enters into the right half-plane, i.e., the harmonic balance method shows, for certain values of the relay element's hysteresis $\kappa_0(\kappa_0 \approx 2)$, an absence of auto-oscillations in the system. In fact, however, there are auto-oscillations for any $\kappa_0 > 0$, as may easily be verified by constructing the relay system's characteristic $J_0(\omega)$. The qualitative difference between the results in the present case, as previously, flows from the essential difference of the auto-oscillation form from a harmonic (sinusoidal) one. In this case one can also use the suggested method of determining the auto-oscillation parameters. It is necessary to compute $J_2(\omega)$ on the basis of (10):

$$J_2(\omega) = -k_p k_0 \left\{ \frac{\pi}{\omega^2 T_0^2} \frac{1}{1 + \operatorname{ch} \frac{\pi}{\omega T_0}} + j\left(\operatorname{th} \frac{\pi}{2\omega T_0} - \frac{\pi}{\omega T_0} \frac{1}{1 + \operatorname{ch} \frac{\pi}{\omega T_0}} \right) \right\},$$

after which Eq. (7) is used for constructing $F(j\omega)$. Figure 6,a shows $J_2(\omega)$ and $F(j\omega)$ (where we have taken $k_0 = 1$) and, on Fig. 6, b, $F(j\omega)$ and $J_0(\omega)$ are shown. As is clear from Fig. 6, b, the difference in the determination of the auto-oscillation frequency ω_0 for any κ_0 is very small.

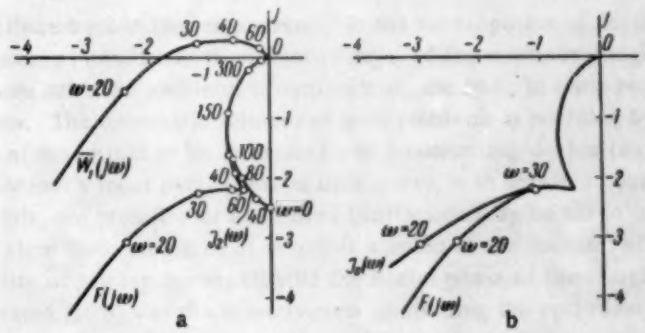

Fig. 6

SUMMARY

1. The harmonic balance method does not give reliable results for the group of relay systems with lagged feedback.

2. The suggested method of taking the exact characteristic of the lagged feedback into account allows one to make direct use of the linear portion's frequency characteristic in investigating relay systems with lagged feedback.

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A MULTICHANNEL AUTOMATIC OPTIMIZER FOR SOLVING VARIATIONAL PROBLEMS

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The block schematic, as well as several functional schematics of the component blocks, are described for an electronic variant of a 12-channel optimizer. Results of testing it as an automatic approximater of a given function of time are provided. A search method is given when "false" extrema, of the "ridge" and "saddle" types are present.

It is well known that there is great interest currently in the development of devices which permit the finding (and retaining, if the external conditions thereafter change) of the most advantageous mode of operation of productive aggregates. There arise the problems of synthesizing the best, in some sense, control system and of best adjusting the aggregates. The automatic solution of such problems is provided by automatic optimization systems [1]. These consist of the object to be optimized and a computing device (an automatic optimizer) which carries out a search of the object's input parameters in such a way as to obtain an extremal of the possible values of some indication. With this, the presence of additional limitations may be taken into account [1]. It was shown [2] that the most efficient algorithm (program) of search is a union of the methods of steepest descent and gradient, this giving the possibility of joining the rapidity of the initial phase of the search with the accuracy of its final phase. Also demonstrated [1, 3] was the effectiveness of dividing the optimizer into two basic blocks, namely, an operational (analog block) and a controlling (digital) block, and a number of simple schemes for the controlling block, implementing the chosen search algorithm, were developed [3]. This paper is a further development of the work cited, and, in particular, is devoted to the description of, and test results for, an electronic variant of a 12-channel automatic optimizer.

Short Description of the Block Schematic

The instrument to be described (Fig. 1) is designed to find an extremum of a function of n variables ($n \leq 12$) given m limitations ($m \leq 4$). It is assumed with this that the function whose extremum is to be found, $Q(x_1, \ldots, x_n)$ possesses one exact extremum, and on its form, as on that of the limiting functions $H_j(x_1, \ldots, x_n)$, there is imposed the condition that the first-order discontinuities (if there are such in the function) do not exceed a given magnitude [1]. The basic task of the optimizer variant to be described is to operate with high-speed models (frequency of solutions up to 30-50 cycles, and thereafter up to 100 solutions per second) while all its elements are to be purely electronic. Certainly, the variant to be described can operate with the ordinary models (for example, model EMU-6).

The search algorithm used in the optimizer is capable of seeking extrema of functions of, generally speaking, arbitrary form.

In fact, the method of search is based on the exploration of the neighborhood of the working points, $x_1, \ldots, x_n$, by means of small tentative increments (steps) of all parameters in turn, and an extrapolation of further motion on the basis of the exploratory data. The results of the explorations are the magnitudes of the

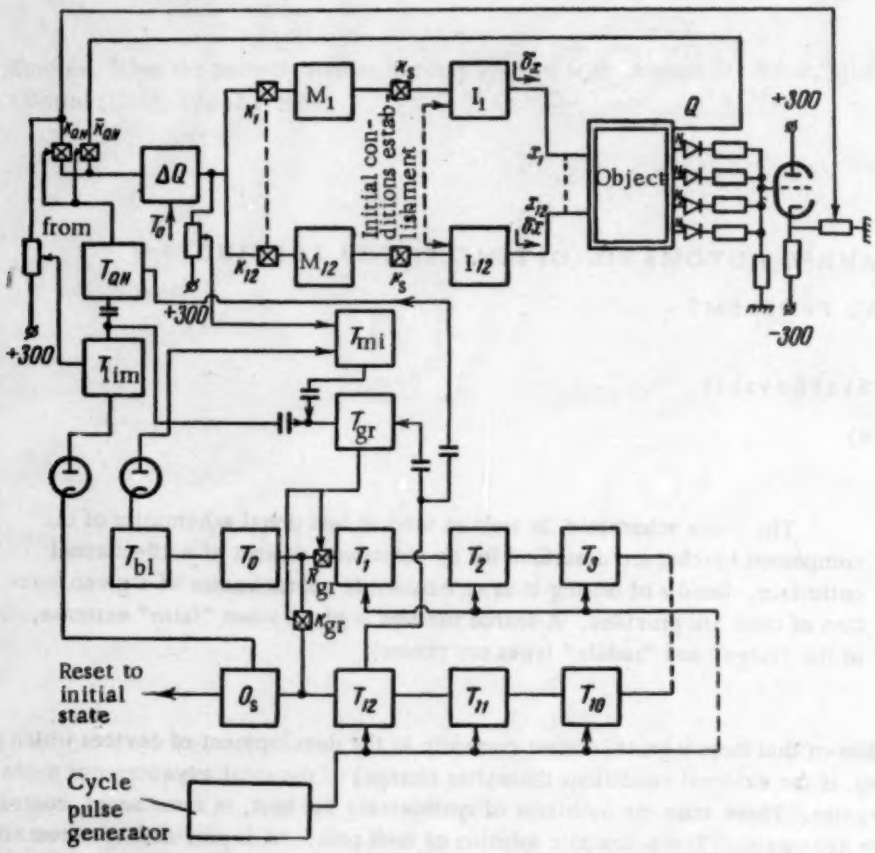

Fig. 1. Block schematic of the 12-channel optimizer

partial derivatives (in practice, the first partial derivatives) which permits the making of a prognosis of the most effective temporal motion of the working step (directed along grad Q). The search process consists of an alternation of exploratory operations (gradient determinations) and working steps in the direction of the gradient, where the order of alternation depends on the search method employed. The instrument consists of two basic blocks, namely, an operational (analog) block and a controlling (digital) one.

The purpose of the operational block is to carry out the determination of the magnitudes of the partial derivatives, to remember them and to carry the working steps for each independent variable, the steps being proportional to the corresponding partial derivatives. In accordance with this purpose, the operational block has 12 parallel channels, at whose common input is located the ΔQ device which determines the first difference of two values of the input signal. Each channel consists of a memory device $M_i (i = 1, \ldots, 12)$ and an integrating device I_i. In addition, there is a determination circuit which gives the output voltage of integrator I_i as the tentative increment δx at the proper moment of time. The outputs of the integrator channels are the object's inputs. The outputs of the object of optimization are the function Q whose extremum is sought, the limiting functions H_1, H_2, H_3, and H_4 and the total limiting function H_0 [1].

The channel elements, as well as each channel with the ΔQ block, are joined by means of electronic keys.

The operation of the apparatus is carried out with a definite succession of closings at certain times (cycle times) of the corresponding keys (or groups of keys) in the operational block. With this, the given order of switching corresponds with a definite order of the operations to be carried out on the quantities Q, H_j and x_i, i.e., is determined by the search algorithm. Thus, for example, closure of key k_0 in the ΔQ block leads to the remembering, in this block's memory cells, the value of $Q(x_{10}, \ldots, x_{120})$. Closure of k_i (k_0 open) with simultaneous application of δx_i leads to the remembering in M_i of the i'th partial derivative. Closure of all k_2 keys leads to the change of x_i by the quantity Δx_i, proportional to $\partial Q / \partial x_i$. The sequency of operations is described in detail in [1, 3] and also in [4, 5].

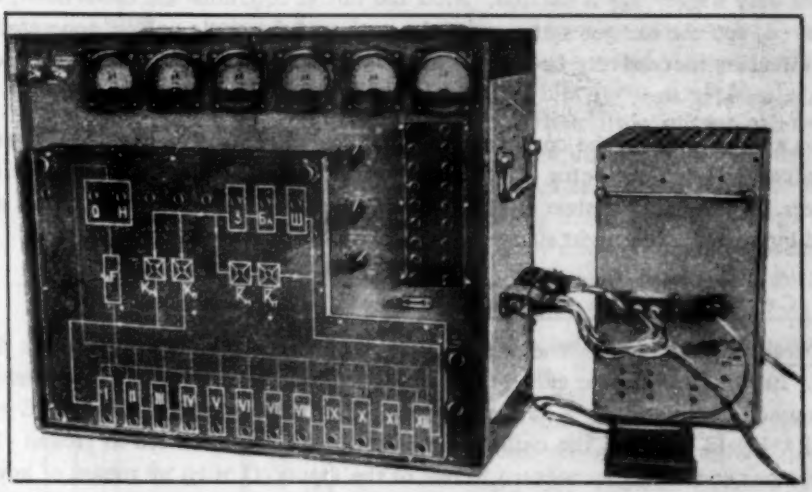

Fig. 2. Photograph of the 12-channel optimizer.

To implement the necessary order of switching (operations), the controlling block is used, several block schematics of which are given in [3]. In the variant described here, the controlling block is constructed on the basis of a refinement of the "two-loop ring" circuit [3] where, as the ring elements, triggers (flip-flops) were chosen. The circuit is a two-loop closed chain (register) of triggers (Fig. 1, near the bottom), each of which is controlled by the corresponding key (or group of keys) in the operational block. If the excited state is induced in one register trigger then, at each following cycle (clock time), the excited state will transfer to the succeeding trigger. Thus, it will circulate in a closed ring either in the large loop of $T_0, T_1, \ldots, T_{12}, O_s, T_{bl}$, depending on the state of the keys k_{gr} and $\bar{k}_{gr}$.

Since each register trigger is controlled by the corresponding keys in the channel (trigger excitation corresponds to key closure), the propagation of the excited state around the large loop, $T_0, T_1, \ldots, T_{12}$ implements the operation of determining the gradient, and localization of the excitation in the small loop T_0, O_s, T_{bl} implements the motion in the direction of the gradient [4].

The circuit refinement is in closing the ring, not immediately, but by means of a reset of the entire register to the null state at the end of a working step. The null state corresponds to the nonexcited state of all the triggers except the blocking trigger, T_{bl}, which is excited in preparing the register for the next cycle. With such a circuit, a random error in the ring (for example, the simultaneous excitation of several triggers) would disrupt optimizer operation only during one cycle of register operation.

The controlling block also includes additional devices for processing the signals necessary for controlling keys k_{gr} and $\bar{k}_{gr}$, operating as a transfer contact and also k_{QH} and $\bar{k}_{QH}$. Among these devices are: 1) the maximum indication trigger T_{mi}, operating together with gradient trigger T_{gr}, which is controlled by keys k_{gr} and $\bar{k}_{gr}$ and 2) limitation detecting trigger T_{lim}, operating jointly with trigger T_{QH}, controlled by keys k_{QH} and $\bar{k}_{QH}$.

The operation of T_{mi} amounts to the reaction to the sign of the increment of the working step as a function of time. Since, with the passage through a particular extremum, the sign of ΔQ changes, this signal is used for sending commands to the gradient determination after several working steps, as this is required in operating by the method of steepest descent. To avoid false operations during the time of gradient determination, when any sign of the derivatives is possible, T_{mi} is blocked during that time by means of trigger t_{bl} in the controlling block. The operation of trigger T_{QH} amounts to the switching of the inputs to the ΔQ block if the boundary $H_0 = 0$ is crossed in some direction or another, whereby there is simultaneously sent a signal to T_{gr}, which switches in the gradient determination. Trigger T_{lim}, which gives the signals switching the instrument input from Q to H_0 and conversely, is also blocked during the time of gradient determination to avoid false switchings during application of tentative steps. More details on circuit operation are described in [5].

To start the optimizer, it is necessary to establish initial conditions, i.e., to establish the output voltages at the outputs of integrating links I_i. For this, by special tumbler switches, the register is places in the initial

(null) state for which only trigger T_{b1} is excited. After the initial conditons are established, the blocking of the initial state is removed, and the excited state, under the action of the cycle pulses, generated by the cycle multivibrator, begins to circulate successively from one trigger to another in an order which depends on the sign of ΔQ at the time of the working step.

Figure 2 shows a photograph of the optimizer with its supply block. The optimizer chassis has the dimensions 670 by 560 by 330 mm, its face being the mounting plate for the individual blocks, which are interconnected by plug-in connectors. When the optimizer is in operation, its state at any moment, and its operation as a whole, are easily inspected by means of the light signals on the mnemonic schematic.

Description of Certain Blocks of the Scheme

1. The ΔQ Determination Block. The ΔQ block determines the accuracy with which the extremum coordinates are found. Its circuit therefore calls for special attention. In the variant described here, the circuit shown in Fig. 3 was used. It is basically a cathode follower with a polystyrene capacitor, C = 1 microfarad, connected in series with the grid circuit. The cathode follower's grid can be shorted to ground via electronic key k_0. The capacitor's other end can be connected to one of the inputs (Q or H) by means of keys k_{QH} and $\bar{k}_{QH}$, operating like a transfer contact. The cathode follower's output is applied to the scale amplifier. When k_0 is closed, the capacitor charges exactly up to the input voltage and, after switching out of k_0, the cathode follower's output voltage equals the difference between the voltage at that given moment and the voltage which existed at the moment when k_0 was switched out [6].

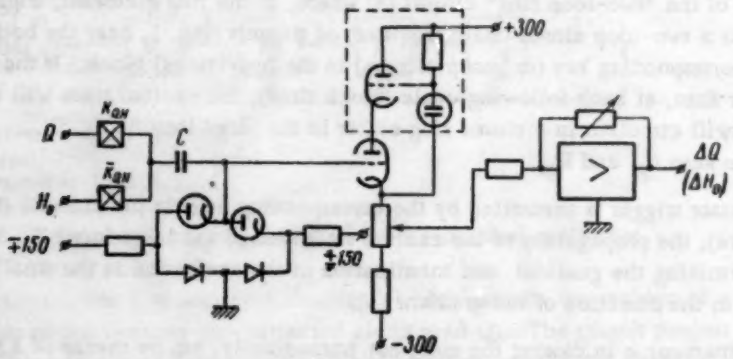

Fig. 3. Circuit for the ΔQ determination block.

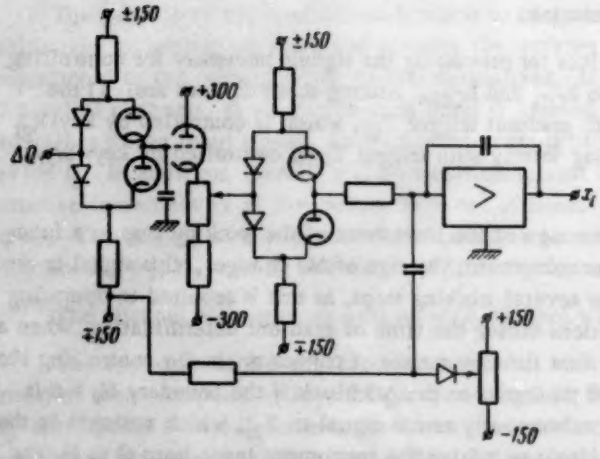

Fig. 4. Block schematic of the channel for variable x_i.

Of fundamental influence on the accuracy of the ΔQ determination is the discharge of capacitor C by the grid current of the cathode follower and the back current of the diode in key k_0. If special measures are not taken, the drift of the output voltage is about 1 volt/min. Better results are obtained by dropping the filament voltage on the vacuum-tube diodes to 5.5 to 5 volts, and by stabilizing the operating mode of the cathode follower by means of a series connected vacuum-tube. By means of these measures, it was easy to obtain drifts not in excess of 0.1 volt/minute on vacuum-tube 6NZP.

2. The Block of the Channel for Variable x_i. (Fig. 4). The memory link is a cathode follower with a capacitance at its input. The capacitance can be charged from the output voltage of the ΔQ block if electronic key k_i, at the link's input, is open. In the closed position of the key, the voltage on the capacitor remains constant.

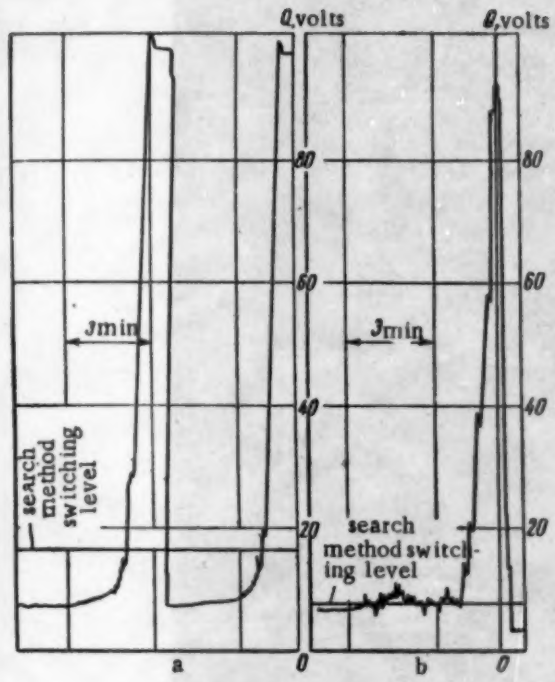

Fig. 5. Diagrams of searches on specimen objects.

The integrator is a three-stage dc amplifier shunted by a capactitive feedback path. The input voltage is applied via electronic key k_3. The tentative increment δx is applied by means of a jump of the controlling voltage on key k_1 through a small capacitor connected to the integrator's input, where the magnitude of the jump can be controlled by a limiting diode. The capacitors in the grid of the memory stage and in the feedback path of the integrator are polystyrene. With this, the drift rate of the integrator's output voltage, when the capacitance in the feedback path is 0.25 microfarads, does not exceed 1 to 2 volts/minute [6].

The initial conditions are established by switching the integrators to the aperiodic link mode of operation. The entire system contains 12 channel blocks.

Bridge-type electronic keys are used in the optimizer, since they allow the number of vacuum-tube diodes to be halved and also possess low forward impedance [6].

Automatic Optimizer Test Results

The preliminary tests of the optimizer were carried out on a reference object which formed an integral part of the instrument. This was a finder of the sum of the moduli of twelve input voltages, i.e., its output was a function of 12 variables with one exact extremum for $U_{in_1} = \ldots = U_{in_{12}} = 0$. Typical search diagrams are shown on Fig. 5, a and b, where the initial steps of the search, in accordance with the method of steepest descent, are shown.

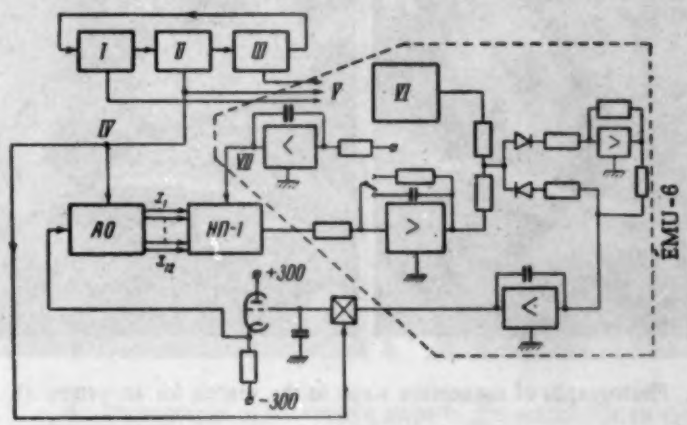

Fig. 6. Block schematic of the set-up for automatic approximation. I is start, II is stop, III is clear, IV is the synchronozing pulses, V is to the control ЭМУ-6, VI is the function supplier, VII is control NP-1.

When close to extremal points, the instrument transfers over to the gradient method, thus permitting one to obtain smaller steady-state oscillations of the function whose extrema are being found. The diagram 5,b shows a case when the optimizer channels were not tuned accurately, but where the transition to the gradient method permitted a smaller error in the determination of Q_{min} to be obtained than in the case when the method of steepest descent was used.

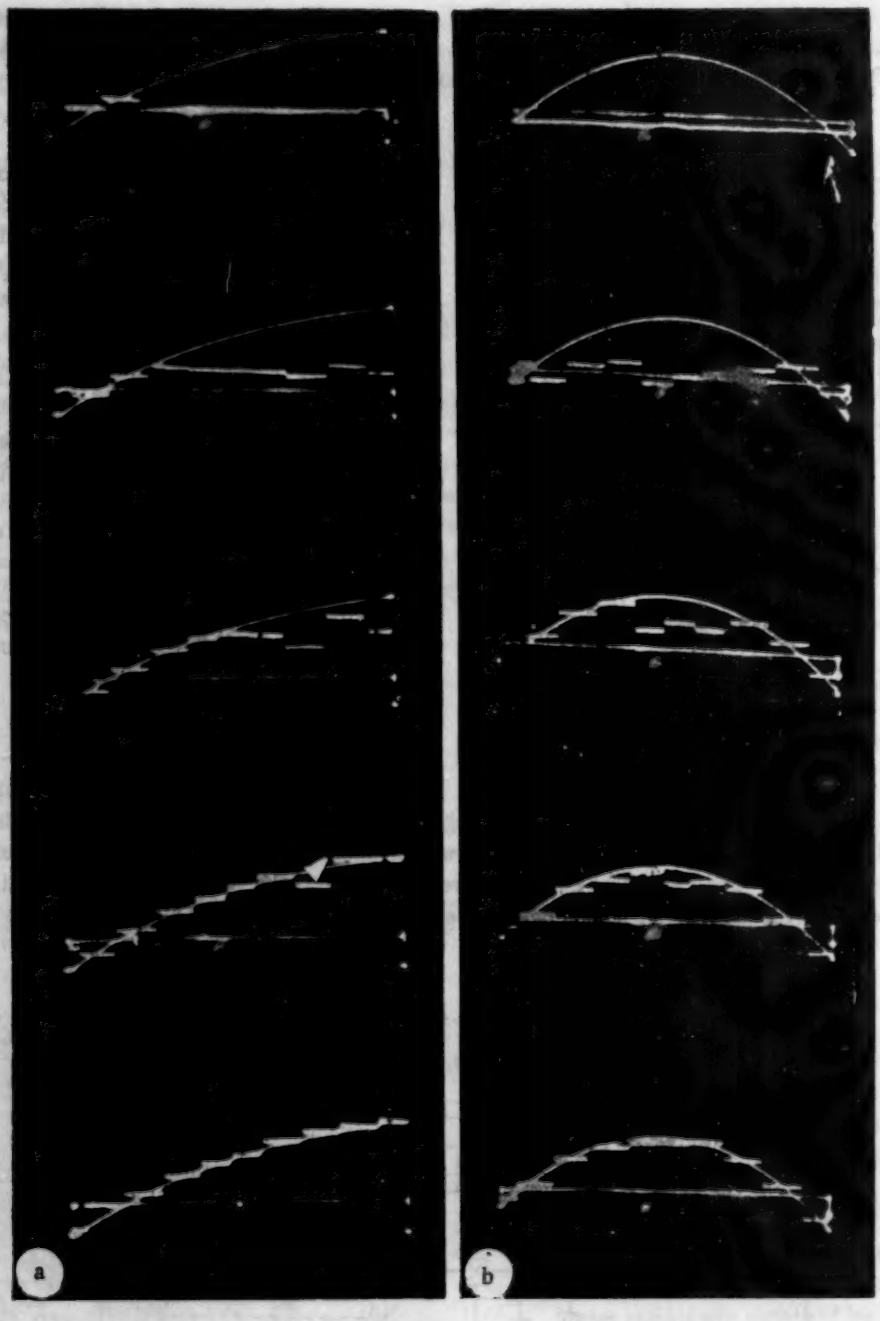

Fig. 7. Photographs of successive steps in the search for an extremal.

As a test problem for the automatic optimizer when operating with an electronic model, we chose the problem of the automatic approximation of a given function of time. In other words, the problem was to construct by means of the optimizer, a function of time which, in the sense of some criterion of nearness, would coincide most closely with a given function of time. As the criterion, the integral criterion

$$Q = \int_0^T |U - U^*| dt,$$

was chosen, where A is the interval of comparison of the two curves, U is the given function of time, generated in the model and U* is the approximating curve constructed by means of the optimizer, i.e., $U^* = f(x_1, \ldots, x_{12})$.

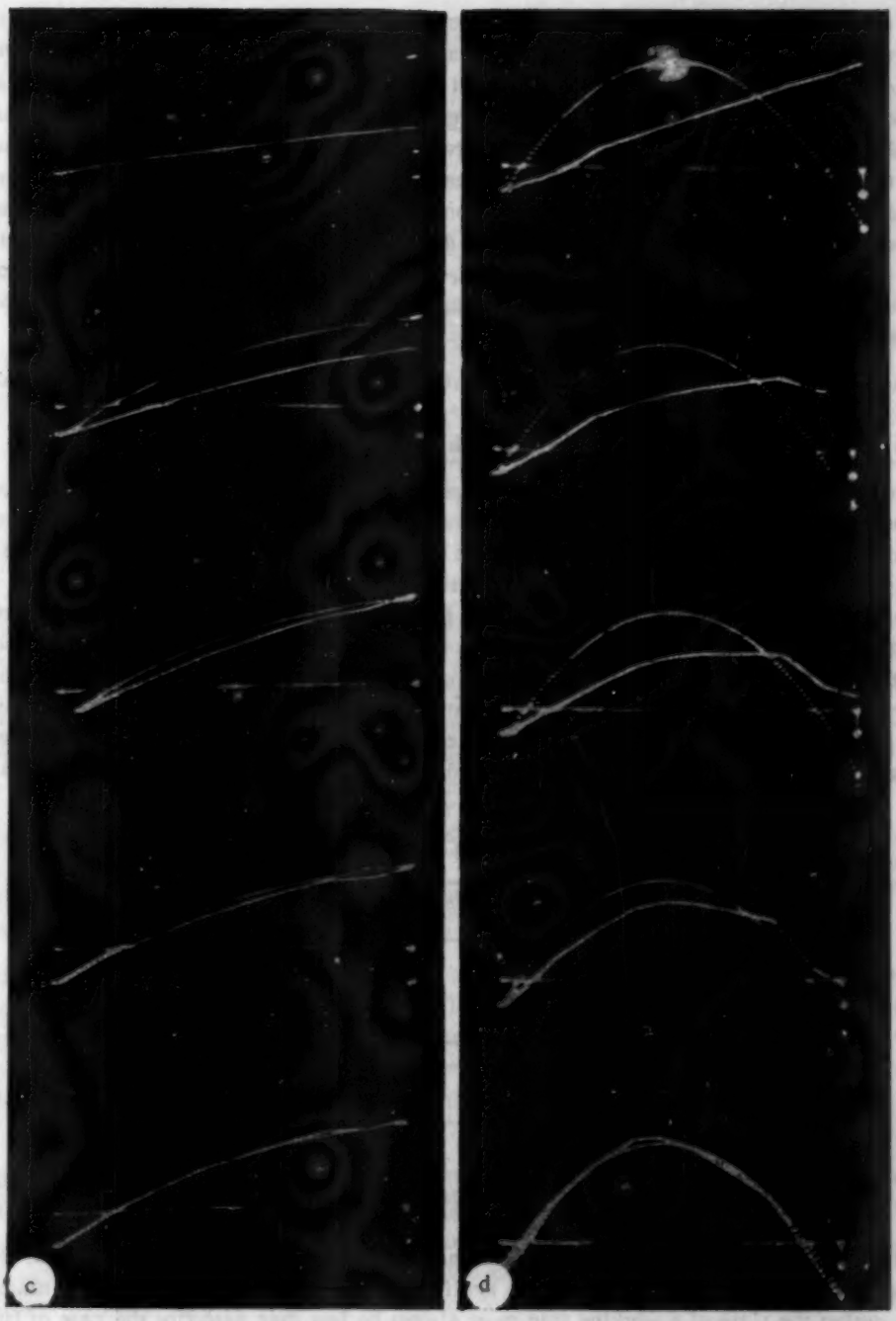

Fig. 7, c, d. Photographs of successive steps in the search for an extremal.

Of course, the condition for the best coincidence of the curves in this sence is that $Q(x_1, \ldots, x_{12}) = \min$.

Figure 6 gives the block schematic of the set-up for the automatic approximation.

For the generation of the approximating function by means of the optimizer we used a nonlinear function generator, developed by L. N. Fitsner, which played the role of a switch, switching the optimizer's output voltage $x_1, \ldots, x_{12}$ to the input of an auxiliary operational amplifier. The auxiliary amplifier could be used either in the scaling or the integrating mode, depending on which one obtained either a stepwise or a linear approximation. Both functions were compared with each other and their difference was applied to the integrator via the block for modulus determination. All the operational amplifiers are blocks of electronic model EMU-6 and are controlled by synchronizing the ring of single flip-flop oscillators "start", "stop" and "clear". This ring gives

signals to the model, to the optimizer and to the memory stage between the model's output and the optimizer's input. The presence of the memory stage eliminates large oscillations of the optimizer's input voltage resulting from the periodization of the solution. It is convenient to observe the approximation process on an electronic oscillograph screen, particularly if the screen has afterglow. For this, both curves are shown on one screen, by means of an electronic commutator, and one can observe how the approximating curve approximates to the continually repeated given curve. Figure 7 shows the successive stages in the approach of the two curves, the figure being obtained by photographing the oscillograph screen. Figure 7,a and b shows the case of a stepwise approximation to exponents and parabolas, while Fig. 7,c and d shows the case of a linear approximation to these same curves. As in clear from the photographs of the curves, the maximum error provided by the approximation (in the case of the linear approximation) was about 4.5% in the case of the exponential curve and about 8% in the case of the parabola.

Thus, the test results show that the process of convergence exists for any form of the function Q.

Certain Questions Regarding Search Method

The most important characteristic of the search process is the time taken to find the solution (certainly, with the necessary accuracy), particularly when there are limitations present and the search process is markedly retarded due to the zig-zag nature of the motion of the representative point about the boundaries [1]. In this connection, a variation of the search methods in the presence of limitations is worthy of attention [7]. Here, the extremum is sought, not for the function Q, but for the function $Q + \alpha H_0$, where α is some proportionality factor. Certainly, in domains free of limitations, $H_0 = 0$. Let one boundary H_j be transgressed and immediately there is a counteracting signal, whereby only the component of Q normal to the boundary is compensated. If α is so chosen that the normal component is exactly compensated, then the condition for an extremum lying on the boundary will be that grad $(Q + \alpha H_0)$ equal zero. It is easily seen that motion tangent to the boundary occurs both in entering a boundary zone and in leaving it, thanks to which the search process is significantly accelerated.

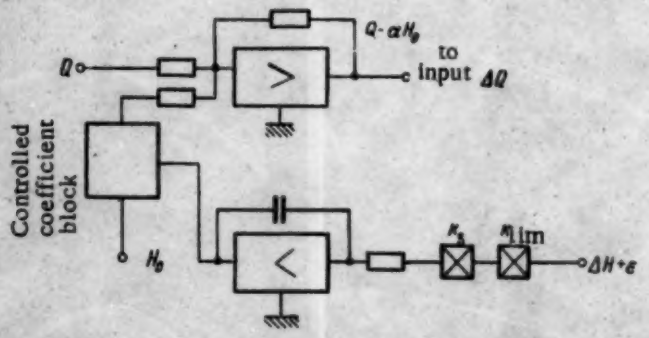

Fig. 8. Circuit for the automatic control of the coefficient α.

It should be stated that, since Q nor H_0 is known beforehand, neither is the coefficient α. However, the choice of α can be made during the search process itself. Indeed, in a boundary region, the magnitude of the increment ΔH_0 during the time of a working step must always be less than zero (for the case of limitations of the type $H_0 \leq 0$) and, generally speaking, the magnitude is small, in order to avoid large oscillations at a nominal minimum point. Thus, control of α may be implemented by means of the simple circuit shown in Fig. 8.

Here a summer of the quantities Q and H_0 is located at the optimizer's input, where H_0 is applied to the summer's input via a block of the controlled coefficient, for example, the product block. The input which controls the block of the coefficient to be controlled is applied from the integrator, at whose input the quantity $\Delta H + \epsilon$ is applied via the two series-connected keys k_s and k_{lim}. These keys are controlled by the single flip-flop step oscillator and, respectively, the limitation indication trigger.

The circuit operates only when k_s is closed, and attempts to maintain $\Delta H = -\epsilon$, where ϵ is a given size of step for H_0 which guarantees, on the one hand, that the boundary zone will be left and, on the other, that there will be small oscillations on the boundary at the nominal extremum point.

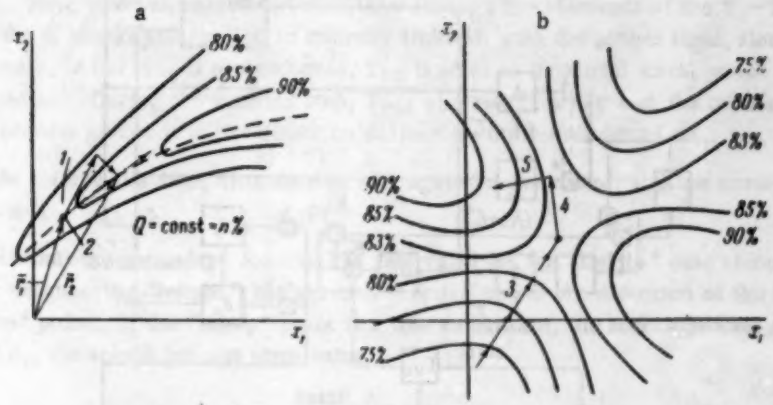

Fig. 9. The cases of "false extrema" (a function of two variables. 1) Excitation; 2) direction of gradient; 3) line of gradient; 4) tentative excitation; 5) direction of search.

Very important also is the question of the class of functions which can be optimized by the instrument. There has already been mentioned in the literature of cases of "false extrema" [8]. Figure 9 shows these cases of "ridges" and saddles" for the simplest case of a function of two independent variables. The function is given by means of lines of equal function values. In both cases, the determination of the components of the gradient by means of providing tentative steps along the coordinate axes leads to zero partial derivatives at the "ridge" points and the "saddle"vertices [8]. However, an extremal value is not found in either case.

These cases differ from one another by the character of the stability of the positions found, namely, in the of a "saddle", the position of the working point will be unstable, and a disturbance of its position, which can be given in order to test the validity of the extremum found, leads to a departure from the "saddle"point towards one of the "humps".

In the case of a "ridge", providing a disturbance after a stable position on the "ridge" is reached, leads to the attainment of another point of this same "ridge". This circumstance can be used for the indirect determination of the direction and magnitude of the gradient. Indeed, since the first stable point may be defined by the radius vector r_1, and the second by the radius vector r_2, the direction of the ridge line is given by the vector $\Delta r = r_2 - r_1$, where the direction of the gradient along this line will be indicated by the sign of the difference $Q_2 - Q_1$.

Thus, the automatic optimizer can carry out verification of the extrema found without making use of the Box-Wilson method [8] with its enormously cumbersome procedure of applying 2^n combinations of tentative increments. To do this, it is only necessary, when the working point has reached a position where $\partial Q/\partial x_1 = \ldots = \partial Q/\partial x_n = 0$, to remember the coordinates of the extremum thus obtained, and then to translate the working point in some arbitrary direction which does not coincide with the direction of approach to the extremal point. Thereafter, the regular search process is carried out, as a result of which a stable state will again be attained. If the coordinates of the new stable state coincide with the remembered coordinates, the search should be discontinued, since the extremum will be a true one. If the coordinates do not coincide, it is necessary to carry out a search along the line of a vector $\Delta r = r_2 - r_1$ and in the direction given by the sign of $\Delta Q = Q_2 - Q_1$, and when a new stable state is reached by means of the regular search procedure, the applying of a disturbance should again be repeated. The procedure just described should be followed until the coordinates of two or three successively found stable positions coincide.

Figure 10 gives one variant of the circuit for realizing the algorithm just described.

In addition to the ordinary block schematic of the optimizer, shown in the right side of Fig. 10 in simplified form, there are n + 1 blocks, Δ, for determining the differences of the coordinates x_i of two stable states and the values of the function Q which correspond to them. The outputs of the Δ blocks, via keys k'_i, are connected to a common input which, either directly or via an inverter, can be connected to the output of block ΔQ

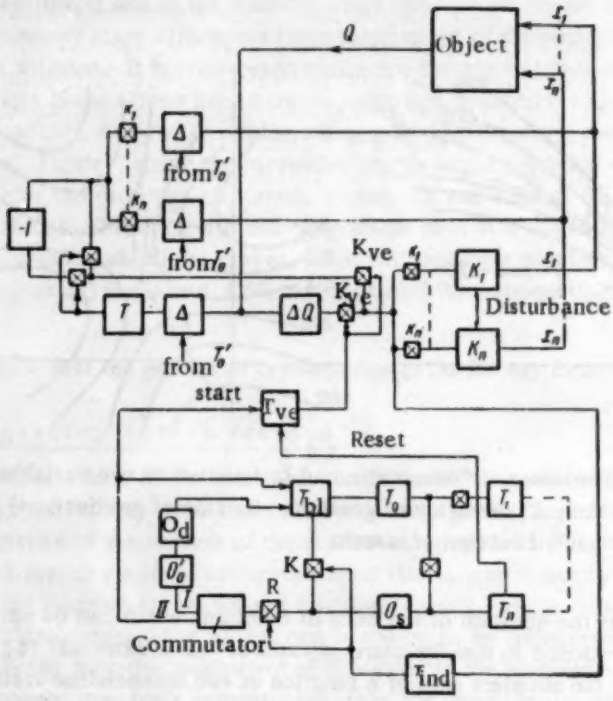

Fig. 10. Simplified block schematic of the optimizer with a device for verifying the validity of the extrema found.

through one of the verification keys $\bar{k}_{ve}$. Keys k_{ve} and $\bar{k}_{ve}$ operate like a transfer contact. Keys k_i' at the blocks' outputs are controlled in parallel with the keys k_i in the channels. The inclusion, or noninclusion, of the inverter is determined by the sign of $\Delta Q = Q_2 - Q_1$. The keys which control the memories in the Δ blocks are controlled by single flip-flop oscillator O'_0 which is added to the controlling block.

In addition, a two-position commutator, shown in the left part of Fig. 10, is added to the controlling block, the purpose of this device being the commutation of pulses which pass through key $\bar{k}$, sending these pulses either through the chain of single flip-flop oscillators O'_0 and O_d and thereafter to T_{bl}, or directly to T_{bl} with a simultaneous reset of T_{ve}. Pulses can be applied to the commutator only in case a stable state has been reached. The attainment of an extremum (false of valid) is fixed by the extremum indication trigger T_{ind} which is connected to the output of the ΔQ block.

Verification of an extremum's validity occurs in the following manner.

After the stable state has been reached (ΔQ is very small during the working step), T_{ind} switches keys $\underline{k}$ and $\bar{k}$ in such fashion that the pulse from the single flip-flop step oscillator is applied now, not to T_{bl} (more exactly, not to the reset of the register to its zero state), but to output I of the commutator. After the pulse is applied, the commutator prepares for the following pulse from output II.

The impulse from output I excites the single flip-flop oscillator for remembering the coordinates of the extremum reached, O'_0, and then the single flip-flop disturbance oscillator O_d, which switches some previously established voltages to the inputs of the channel integrators. After the disturbances are applied, T_{bl} and then T_0 are excited, where T_0 resets the output voltage of the ΔQ block to zero and, since the magnitude of the most recent working step was small, the determination of the gradient occurs (Cf., the description of the optimizer's block diagram). After the determination of all the partial derivatives, the working step follows and a large magnitude of ΔQ appears since, as a result of the disturbance, the working point moved away from the stable state. Therefore, T_{ind} opens key $\bar{k}$ and closes k, after which the search process described at the beginning of this paper transpires.

When the new stable state is reached, T_{ind} again operates and, and after the next working step, a pulse is applied to verification trigger T_{ve}. This trigger, by means of keys k_{ve} and $\bar{k}_{ve}$, switches the channel inputs from

the output of the ΔQ block to the outputs of the Δ blocks, either directly or via the inverter, depending on the sign of $\Delta Q = Q_2 - Q_1$. Now, when an excitation circulates through the elements of the $T_1 - T_{12}$ register, the Δx_i from the outputs of the Δ blocks are applied to memory links M_i with the proper signs, since keys k_i and $\bar{k}_i$ are switched simultaneously. After Δx_n is remembered, T_{ve} is reset to its initial state, which corresponds to the ordinary search procedure. During the working step, T_{ind} also switches key $\underline{k}$ to the ordinary ring. After this, the ordinary search process proceeds in the direction defined by the remembered Δx_i.

After the stable state is reached, disturbances will again be generated, and the processes of verifying an extremum are repeated.

It is easily seen that the procedure described is also valid for the "saddle" case since, after reaching a vertex of one of the neighboring "humps", the search is carried out in the direction of the line joining the "saddle" point with the "hump" point. If the "hump" point is a true extremum, the search process returns to the same point and $\Delta x_i = 0$, i.e., the search process terminates.

SUMMARY

1. The instrument described may be used for solving by direct methods a broad class of variation problems, both with limitations and without them. The simplest examples were given in the paper.

2. The number of independent variables (12, in our case) can, if necessary, be greatly increased without undue complication of the instrument.

The author wishes to thank A. A. Fel'dbaum for this scientific direction, and A. V. Kalinina, A. B. Shubin and I. N. Bocharov for taking part in the adjustment and testing of the instrument.

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PNEUMATIC SWITCHING CIRCUITS

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(Moscow)

Descriptions are given of apparatus used in constructing pneumatic switching circuits. Methods are demonstrated of realizing by pneumatic means the elementary logical functions and a one-cycle delay function. It is also shown that any finite automation can be constructed by means of pneumoautomata.

The circuits are given for several pneumatic devices which implement analog-to-digital conversions. Examples are given of the application of pneumatic switching circuits in industrial pneumoautomatic devices.

INTRODUCTION

Contemporary systems for the automatic control of productive processes contain, as their most important constituent parts, various computing devices and devices which implement certain logical functions (switching circuits).

In industrial automation, a tremendously extensive use is made of pneumatic apparatus; this is related both to the superior features of pneumatic executive mechanisms and to the fact that this apparatus is simple, reliable and, by its nature, explosion- and fireproof. There are many factories where all the automation is effected by pneumatic means. If, in the control systems of such factories, it is necessary to use computers or switching circuits and, if these latter need not, under actual production conditions, be very fast-acting, then it would certainly be eminently reasonable to use pneumatic variations of these devices and circuits.

In gauging the domains where it would be advantageous to use pneumatic computing devices and switching circuits, one must also take into account that, in the first place, in chemistry, petroleum processing, thermotechnics, thermoenergetics, metallurgy and many other branches of industry one can use computing devices and switching circuits with speeds of operation which are fully attainable by pneumatic means and, in the second place, that pneumatic apparatus is in principle efficient for high-temperature operation.

In the Laboratory for Pneumohydraulic Automation of the IAT AN SSSR, work has been carried out on the construction of pneumatic analog computers and pneumatic switching circuits. The present paper shows how switching circuitcs can be realized by pneumoautomatic means*.

At the base of the pneumatic switching circuits to be described here is the same relay-contact principle which is used in constructing electromechanical switching circuits [1]. In the pneumatic variant this principle also leads to devices which consist of circuits with relays connected in them, but in this case both the circuit itself and the relays are pneumatic. The signals on which the switching circuits operate also have a pneumatic nature here. In the circuits under consideration, the signal levels will have the following correspondence with the pressure of the compressed air: a signal with the symbol 1 corresponds to a pressure of 1 atmosphere, a signal with symbol 0 to a pressure of 0 atmospheres.

* The results of this work were present on September 26, 1958 at a conference on the application of digital (discrete) automatic systems. Also participating in this work were L. I. Rozonoér, A. A. Tagevskaya, N. V. Grishko, T. K. Morozova, P. A. Lebedev and M. M. Sharafetdinov.

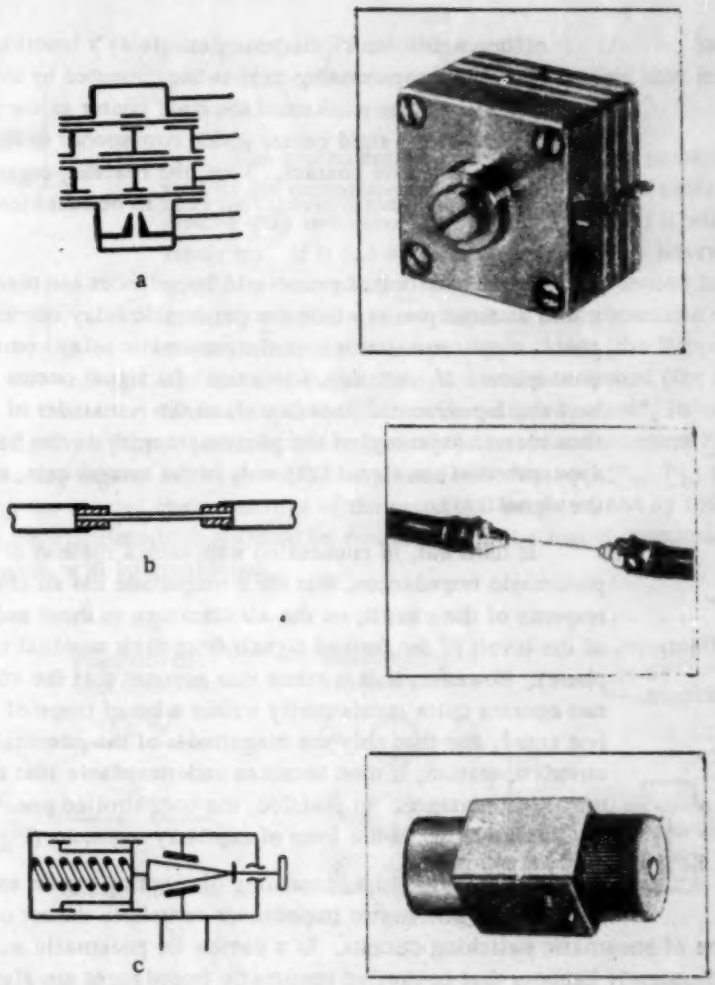

Fig. 1. Apparatus used in constructing pneumatic switching circuits. a) A pneumatic relay; b) an uncontrolled pneumatic valve; c) a controlled pneumatic valve.

With the method chosen for forming the signal levels in the pneumatic switching circuits being considered, the pneumatic circuits which form them will contain, in addition to pneumatic relay, also uncontrolled pneumatic impedances.

A pneumatic relay (Fig. 1,a), just as a relay used in electromechanical devices, consists of two basic parts: a reacting organ and contact (contacts). The pneumatic contact is realized in the form of a nozzle-flapper pair. This element, established in the pneumatic circuit, constilates a local impedance whose magnitude depends on the distance between the flapper and the nozzle and may be varied from zero, when the flapper is removed a sufficient distance from the nozzle, to infinity, when the flapper is moved right next to the nozzle. If the nozzle, when controlled, can assume only these two positions, then the nozzle-flapper element will implement the function of a contact, i.e., will either completely open the pneumatic circuit or close it moreover, without the introduction of additional impedances in the circuit, in practice. In the future, when a nozzle-flapper element is used as a contact, we shall say that the pneumatic contact is closed when the flapper is remote from the nozzle, and that it is opened when the flapper completely covers the nozzle.

The reacting organ of the pneumatic relay consists of three plane membranes. They are connected axially to a rigid center whose movements are limited by two bearings, while the membrance peripheries are fixed in the frame. In the dead-end chambers which separate the membranes the command pressures are applied, these causing the appearance of some total force which acts on the rigid center along its axis. This force shifts the rigid center towards the bearing located on the side to which the force is directed.

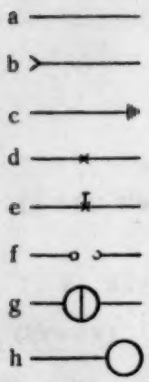

Fig. 2. a) A pneumatic circuit path; b) the supplying end of a pneumatic circuit path; c) the "grounded" end of a pneumatic circuit path; d) an uncontrolled pneumatic valve; e) a controlled pneumatic valve; f) a pneumatic contact; g) a pneumatic relay's reacting organ; h) a pneumatic capacitance.

The connection of the pneumatic relay's reacting organ with the pneumatic contacts appertaining to it is implemented by using one of the bearings which limit the motions of the rigid center as the nozzle and by using the butt end of the rigid center which corresponds to this bearing as the flapper of the pneumatic contact. Thus, the reacting organ of the pneumatic relay and its pneumatic contact turn out to be combined in the physical design.

The uncontrolled pneumatic impedances are placed in the circuits either at those places where the pneumatic relay connects with the 1-atmosphere supply trunk, or where the pneumatic relay communicates with the atmosphere. If, with this, formation of a signal occurs in a portion located between a pneumatic impedance and the remainder of the pneumatic circuit, then closure (opening) of the pneumatic relay in the first case leads to the appearance of the signal 0(1) and, in the second case, to the appearance of the signal 1(0).

It turns out, in connection with such a method of using uncontrolled pneumatic impedances, that their magnitude has an effect on the speed of response of the circuit, on the air discharge in them and on the deviations of the levels of the formed signals from their nominal values (0 or 1 atmosphere). However, if it is taken into account that the circuits to be realized can operate quite satisfactorily within a broad range of the characteristics just noted, and that only the magnitudes of the pneumatic impedances affect circuit operation, it then becomes understandable that this element is of modest importance. In practice, the uncontrolled pneumatic impedances are implemented in the form of capillary segments (Fig. 1,b).

Pneumatic relays, consisting of reacting organs and pneumatic contacts and constant pneumatic impedances constitute the set of basic apparatus necessary for the construction of pneumatic switching circuits. In a device for pneumatic automation which contain switching circuits, it frequently happens that controlled pneumatic impedances are also used. The circuit for such an element is shown in Fig. 1,c.

The connection of the individual pneumatic relays and pneumatic impedances in one device (the working mounting of the circuit) is done on special plates. The pneumatic relays and the controlled pneumatic impedances are mounted on the surfaces of these plates, and all communications (pneumatic circuits) occur inside them.

Naturally, a pneumatic relay-contact apparatus, being analogous to the corresponding electrical apparatus, will possess the same functional capabilities. It is shown below in what ways pneumatic relay-contact apparatus should be used to create devices which bear the generic designation of "finite automata" (Cf., for example, [2]). We first give methods of constructing devices which realize the elementary logical operations. On the basis of these devices we can construct circuits in which operations are implemented which are defined by logical functions of any complexity, the so-called primitive circuits. We then give the circuit for the most important link in finite automata, namely, the delay element. It is then shown how the so-called nonprimitive circuits may be constructed on the basis of the devices in which the elementary logical operations necessary for the construction of primitive circuits are implemented and of the delay elements.

In addition, we provide certain information on the construction of analog-to-digital devices which are based on the same apparatus. In concluding this work, we provide examples of the use of pneumatic relay-contact apparatus in the circuits of certain devices which are of important practical value.

The basic content of the material given below is contained in the circuit diagrams which, for economy of space, are accompanied by short explanations only.

Notation and Symbolism

For the graphic representation of pneumatic switching circuits we use the notation enumerated on Fig. 2. In addition to these designations we also use a special symbolism which represents the method of connecting the

TABLE 1

P_a	0	1
$x = P_a$	0	1

TABLE 2

P_b	0	1
$x = \bar{P}_b$	1	0

pneumatic relay's reacting organ in the circuit. Essentially, this symbolism is a variant of the corresponding symbolism used in the theory of electrical relay-contact circuits [1].

The fundamental variable quantity in pneumatic circuits is the magnitude of the compressed air pressure. In those cases, when this variable can assume only two values (0 and 1 atmosphere) it will be denoted by P with subscripts. If it can assume any value in the interval from 0 to 1 atmosphere, it will be denoted by $\underline{P}$ with subscripts. A second important variable is the quantity $\underline{x}$ which defines the state of a pneumatic contact; we shall say that $x = 1$ if the pneumatic contact is closed (the flapper is remote from the nozzle) and $x = 0$ if the pneumatic contact is opened (the flapper covers the nozzle). Variable $\underline{x}$ is the pneumatic relay's "output", its value being determined by state of the pneumatic relay's "input", i.e., by the values of the command pressures $\underline{P}_a$ and $\underline{P}_b$ applied to the pneumatic relay's reacting organ. In the general case the relationship $x = x(\underline{P}_a; \underline{P}_b)$ is established. Its concrete form is determined by the special characteristics of the command pressures and by the method of connecting the pneumatic relay. In the circuits which we shall be considering in the sequel, pneumatic relays, realizing five forms of this relationship, will be considered.

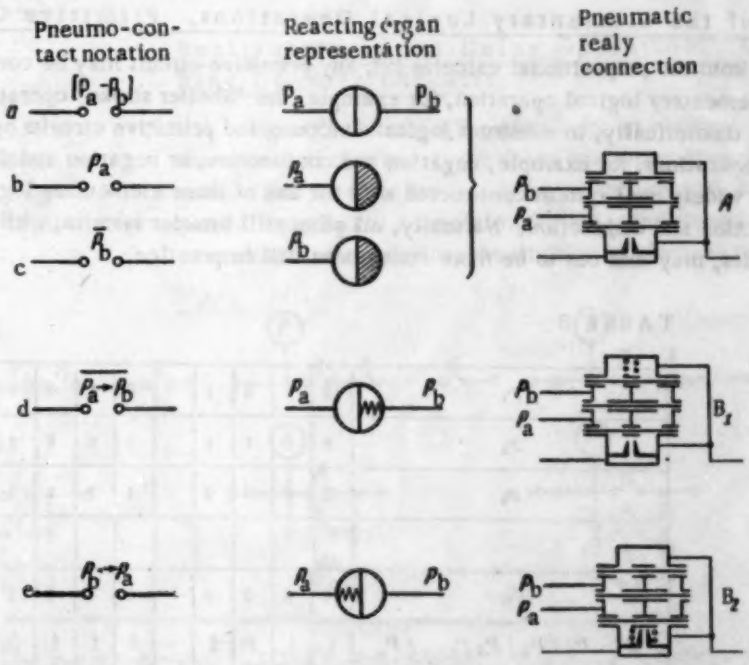

Fig. 3. The relationships realized in pneumatic relays and the corresponding symbolic representations of the reacting organs and pneumatic contacts: a) $x = Sg(\underline{P}_a - \underline{P}_b) \equiv [\underline{P}_a - \underline{P}_b]$; b) $x = P_a$; c) $x = P_b$; d) $x = \overline{P_a \to P_b}$; e) $x = P_b \to P_a$.

a) $x = Sg(\underline{P}_a - \underline{P}_b) \equiv [\underline{P}_a - \underline{P}_b]$. A pneumatic relay which realizes this relationship will be called a receiving relay; it is connected in accordance with circuit A, shown on Fig. 3. The command pressures $\underline{P}_a$ and $\underline{P}_b$ applied to it may vary within the interval of 0 to 1 atmosphere.

b) $x = P_a$ (Table 1). To realize this function, the pneumatic relay is connected in accordance with the same circuit A, but has one input P_a which assumes values of either 0 or 1; the pressure supplied to the second input remains invariant (fixed) and equals approximately 0.5 atmospheres.

TABLE 3

P_a	0	1	0	1
P_b	0	0	1	1
$x = \overline{P_a \rightarrow P_b}$	0	1	0	0
$x = P_b \rightarrow P_a$	1	1	0	1

TABLE 4

P_1	0	1
$P = \overline{P}_1$	1	0
$P = P_1$	0	1

c) $x = \overline{P}_2$ (Table 2). In this case the pneumatic relay is again connected in accordance with circuit A and has one input, P_b; with this, the second input remains "fixed" at the value of 0.5 atmospheres ($P_a \approx 0.5$ atmospheres).

d) $x = \overline{P_a \rightarrow P_b}$ (Table 3). Here, both inputs are "free", i.e., both P_a and P_b are variables which assume the values of 0 and 1. To realize this function, a spring is provided to the reacting organ of the pneumatic relay, as shown on Fig. 3 in circuit B_1.

e) $x = P_a \rightarrow P_b$ (Table 3). This case differes from the previous one by the location where the spring is places (Fig. 3, circuit B_2).

The symbolic representations of the reacting organs and the contact notations shown in Fig. 3 correspond to these five methods of employing a pneumatic relay. The method used to notate the contacts is exhaustive in relation to the information on the methods of connecting the corresponding pneumatic relay.

Realization of the Elementary Logical Operations. Primitive Circuits.

As follows from the propositional calculus [3], any primitive circuit may be constructed with the use of only some one elementary logical operation, for example, the "Sheffer stroke" operation. However, it is more convenient, even theoretically, to construct logical functions and primitive circuits by the use of some two elementary logical operations, for example, negation and conjunction, or negation and disjunction. Even more convenient is the widely used system constructed with the use of three elementary logical operations, namely, negation, conjunction and disjunction. Naturally, all other still broader systems, while providing no new theoretical capabilities, may turn out to be more convenient still in practice.

TABLE 5

P_1	0	1	0	1		0	1	0	1
P_2	0	0	1	1		0	0	1	1
P_3	0	0	0	0		1	1	1	1
⋮									
P_n	0	0	0	0		1	1	1	1
$P = P_1 / P_2 / P_3 / \ldots / P_n$	1	1	1	1		1	1	1	0
$P = P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n$	0	0	0	0		0	0	0	1
$P = P_1 \vee P_2 \vee P_3 \vee \ldots \vee P_n$	0	1	1	1		1	1	1	1

We show below now, by pneumatic means, one realizes the operations of "negation", "replication", "Sheffer stroke", "conjunction", "disjunction","implication", "equivalence" and "logical sum". Such a list of elementary logical operations is known to be sufficient for the construction of logical functions of any complexity and, consequently, any primitive circuit may be constructed on the basis of devices which realize these operations.

All these operations may be divided into three groups:

1) One-input (Table 4) – negation ($P = \overline{P}_1$) and replication ($P = P_1$).

2) n-input (Table 5) – Sheffer stroke ($P = P_1 / P_2 / \ldots / P_n$), conjunction ($P = P_1 \wedge P_2 \wedge \ldots \wedge P_n$) and disjunction ($P = P_1 \vee P_2 \vee \ldots \vee P_n$).

TABLE 6

P_1	0	1	0	1
P_2	0	0	1	1
$P = P_1 \to P_2$	1	0	1	1
$P = P_1 \sim P_2$	1	0	0	1
$P = P_1 \nabla P_2$	0	1	1	0

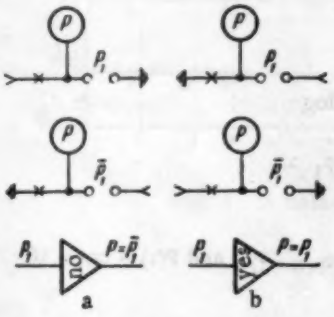

Fig. 4. Circuits of devices realizing the elementary logical functions of one independent variable (Table 4). a) "Negation", "no", $P = \overline{P}_1$; b) "replication", "yes, $P = P_1$.

3) Two-input (Table 6) – implication ($P = P_1 \to P_2$), equivalence ($P = P_1 \sim P_2$) and logical sum ($P = P_1 \nabla P_2$).

The circuits for the devices which realize the operations of the first group are shown on Fig. 4. All these circuits are constructed with the use of pneumatic relays of the type $x = P_a$ or $x = \overline{P}_b$ (Fig. 3), each type of relay having its corresponding variant of circuit construction. The circuits for the devices, realizing the operations of the second group, are shown in Fig. 5. They are constructed with pneumatic relays of the same types. For each operation here, two variants of circuits construction are given: with series and with parallel connections of the pneumatic contacts.

The circuits of the devices which realize the operations of the third group are shown in Fig. 6. These circuits are constructed with pneumatic relays of the types $x = \overline{P_a \to P_b}$ or $x = P_b \to P_a$ (Fig. 3). Here also, each type of relay has its corresponding variant of circuit construction.

On the last three figures, under the circuit which realizes a particular operation, there is given the symbolism for this device.

Realization of the Delay Operation. Delay Lines. Ring Counter Circuits.

Figure 7 represents the circuit for an element which implements the operation of a one-cycle delay (the delay time is one nominal time unit). This element has two independent inputs P_1 and P_t and one output P. Input P_t has a special meaning: the successive appearance on it of ones and zeroes defines the division of physical time into cycles (into

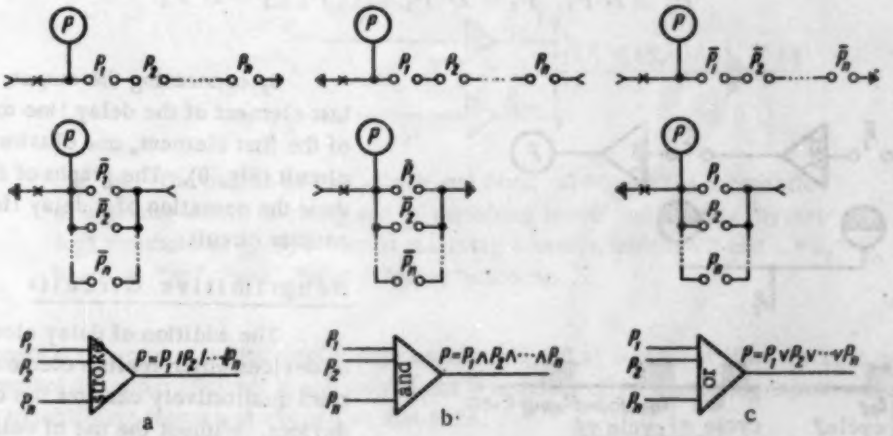

Fig. 5. Circuits of devices realizing the elementary logical functions of n independent variables (Table 5). a) The "Sheffer stroke", $P = P_1 / P_2 / \ldots / \overline{P}_n$; b) "conjunction", "and", $P = P_1 \wedge P_2 \wedge \ldots \wedge P_n$; c) "disjunction", "or", $P = P_1 \vee P_2 \vee \ldots \vee P_n$.

units of nominal time), just as is shown on Fig. 7. For further convenience at each moment of time corresponding to some unit of nominal time we distinguish two successive moments: one, t_{bn}, corresponds to the end of the n'th cycle, and the other, $t_{a(n+1)}$, to the beginning of the (N + 1)'st cycle. In an idealized circuit these two moments coincide. We shall assume that $t_{an} = t_{b(n+1)} + 0$. The delay operation realized by the element under consideration relates its input and output by the relationship $P' = D^1 P_1$ which, with the notation we have

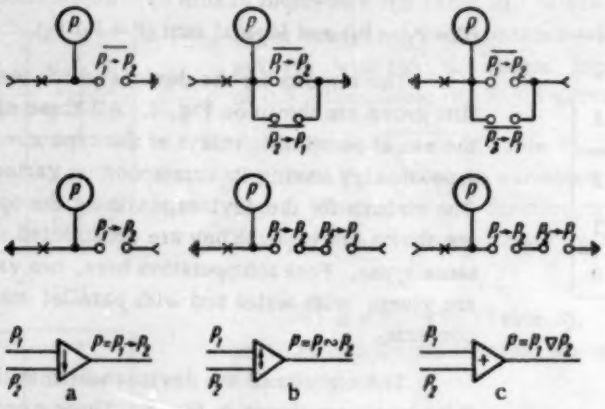

Fig. 6. Circuits of devices realizing the elementary logical functions of two independent variables (Table 6).
a) "Implication" $P = P_1 \to P_2$; b) "equivalence", $P = P_1 \sim P_2$; c) "logical sum", $P = P_1 \vee P_2$.

assumed, has the following content: $P(t) = P_1(t-0)$ if $t = t_{an}$ (with this, $t - 0 = t_{b(n-1)}$) and $P(t) = \text{const}$ if $t_{an} < t < t_{a(n+1)}$.

The graphs of Fig. 7 illustrate the operation of the delay element which, in fact, amounts to the "remembering" of the input quantity during the time corresponding to one cycle. Each cycle may be divided into two parts: in one of them, $P_t = 0$ and, in the other, $P_t = 1$. In correspondence with this, the delay element also consists of two parts: one of them "remembers" during the time when $P_t = 0$, and the other when $P_t = 1$. A series connecttion of n delay elements in one chain forms a delay line (Fig. 8) in which the following operation is implemented:

$$P_2 = D^1 P_1, \quad P_3 = D^2 P_1, \ldots, P_{n+1} = D^n P_1.$$

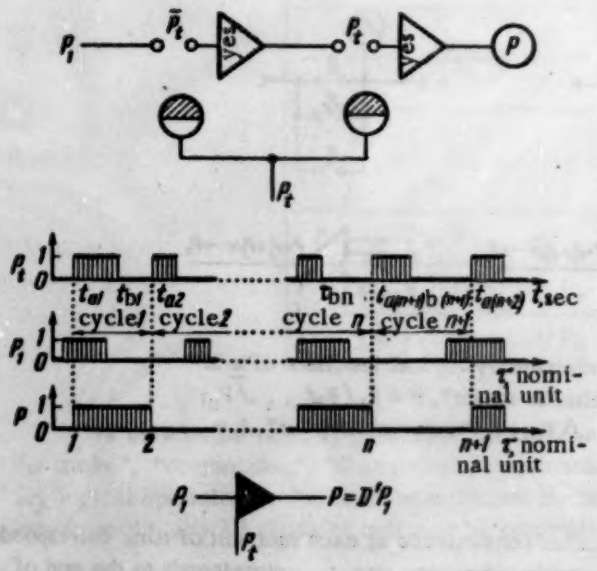

Fig. 7. A one-cycle delay element, $P = D^1 P_1$.

By connecting the output, P_{n+1}, of the last element of the delay line to the input, P_1, of the first element, one obtains a ring counter circuit (Fig. 9). The graphs of Figs. 8 and 9 show the operation of a delay line and a ring counter circuit.

Nonprimitive Circuits

The addition of delay elements to a system of devices which realize elementary logical functions qualitatively changes the capabilities of these devices. Without the use of delay elements in them, only primitive circuits can be constructed, whereas when used with delay elements, they give the possibility of constructing nonprimitive circuits of any degree of complexity, i.e., any finite automata. This is borne out by the following. It is known from the theory that any finite automaton can be constructed from universal elements. As such an element, one may take the model of a nerve cell (neuron) suggested by McCulloch and Pitts [2].

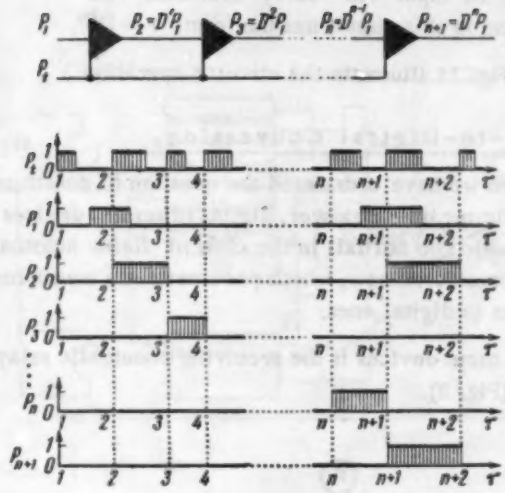

Fig. 8. A delay line.

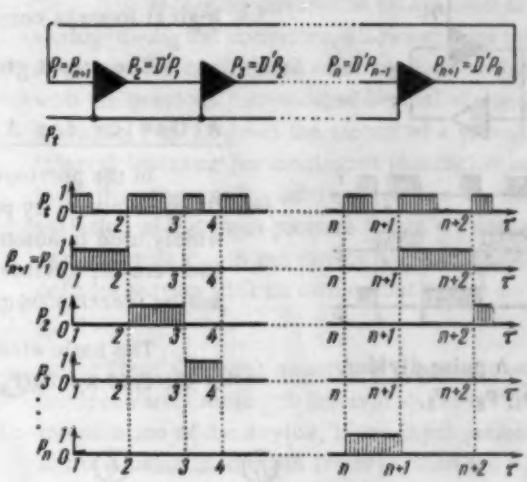

Fig. 9. A ring counter circuit, $P_2 = D^1P_1$, $P_3 = D^1P_2$, . . ., $P_{n+1} = D^1P_n$, $P_1 = P_{n+1}$.

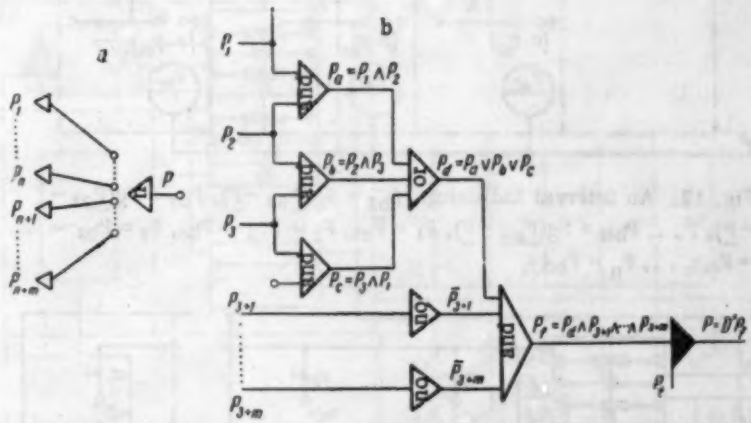

Fig. 10. The neuron of McCulloch and Pitts. a) Simplified schematic of a neuron with n exciting and m inhibiting inputs and a firing (operating) threshold of h; b) a circuit realizing a neuron with n = 3 and h = 2, based on "no", "and", "or and "delay"elements.

The fundamental property of a Pitts-McCullogh neuron (Fig. 10,a) is that, at time t + 1, there appears at its output P the symbol 1 (output excited) if and only if not less than h of its n exciting inputs was excited at time t and none of its m inhibiting inputs was excited at that time.

Figure 10,b shows a neuron with n = 3 and h = 2, its logical formula having the form:

$$P = D^1\{[(P_1 \wedge P_2) \vee (P_2 \wedge P_3) \vee (P_1 \wedge P_3)] \wedge \overline{P}_{3+1} \wedge \ldots \wedge \overline{P}_{3+m}\}.$$

This formula is constructed from the operations of "no" ("not"), "or", "and" and "delay", and the logical formula for any other neuron is also constructed with these same operations. Thus, the capability of realizing these four operations is a condition for the possibility of constructing any neuron and, consequently, any finite automaton. It was shown previously how these four operations are implemented by pneumatic means. This means that it is possible to produce any neuron and any finite automaton by pneumatic means.

As an example of the realization of a nonprimitive circuit, Fig. 11 gives the circuit for a pneumatic pulse divider (a pneumatic trigger). This simplest, but practically important, circuit consists of a "no" element whose

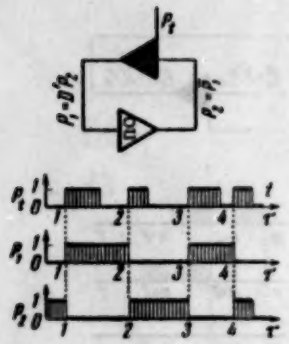

Fig. 11. A pulse divider, $P_1 = D^{\tau}\overline{P}_1$, $P_2 = \overline{P}_1$.

output is connected back to its input via "delay" element. The logical formula corresponding to this circuit has the form: $P = D^{\tau}\overline{P}$.

The graphs given on Fig. 11 illustrate the circuit's operation.

A Device for Analog-to-Digital Conversion.

In the previous sections we have considered the question of constructing finite automata by pneumatic means. However, digital (discrete) devices are widely used in automation which do not fall in the class of finite automata. Such are the devices, to be considered now, which pneumatically transform analog (continuous) quantities to digital ones.

The basic element in these devices is the receiving pneumatic relay of the type $x = Sg(\underline{P}_a - \underline{P}_b)$ (Fig. 3).

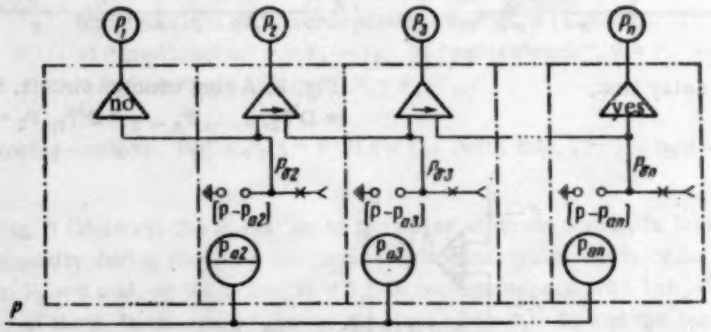

Fig. 12. An interval indicator: $P_{b2} = Sg(\underline{P}_{a2} - \underline{P})$, $P_{b3} = Sg(\underline{P}_{a3} - \underline{P})$, . . . , $P_{bn} = Sg(\underline{P}_{an} - \underline{P})$, $P_1 = \overline{P}_{b2}$, $P_2 = P_{b2} \to P_{b3}$, $P_3 = P_{b3} \to \to P_4$, . . . , $P_n = P_{bn}$.

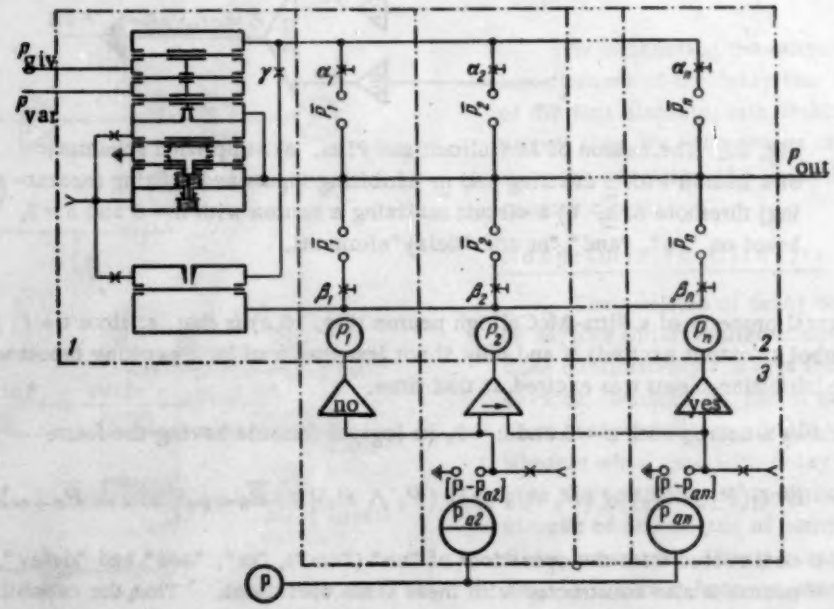

Fig. 13. A pneumatic isodrome regulator with automatic changeover. 1) Pneumatic regulating block; 2) contains valves α and β with their vents; 3) interval indicator.

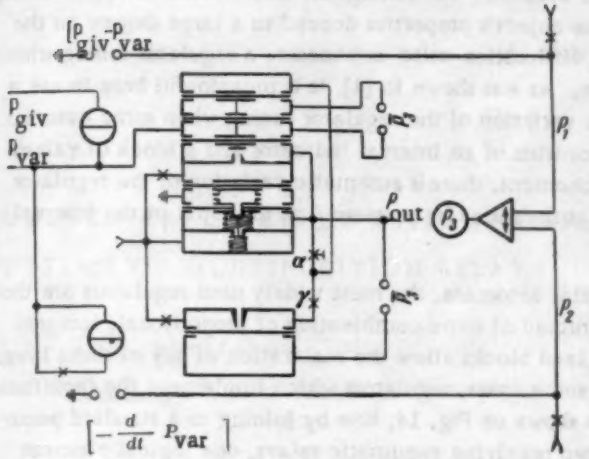

Fig. 14. A semiproportional regulator.

The receiving pneumatic relay, used as an analog-to-digital converter, allows various receiving devices to be constructed when used in conjunction with the previously considered logical element. For example, Fig. 12 gives the circuit of a so-called interval indicator for continuous (analog) quantities. In it, the pressures $\underline{P}_{al}$ subdivide the entire range of variation of the input pressure $\underline{P}$ into $\underline{n}$ intervals; to each pressure $\underline{P}_{al}$ in the device there corresponds a definite section with its own output P_i (we note that $\underline{P}_{al} = 0$).

Thus, the input quantity $\underline{P}$ is put into correspondence with some $\underline{i}$'th interval by the output P_i. In operation of the device, if the input pressure assumes a value in the $\underline{i}$'th interval, then the P_i which corresponds to this interval will have a zero value, and all the remaining sections will have output pressures of unity.

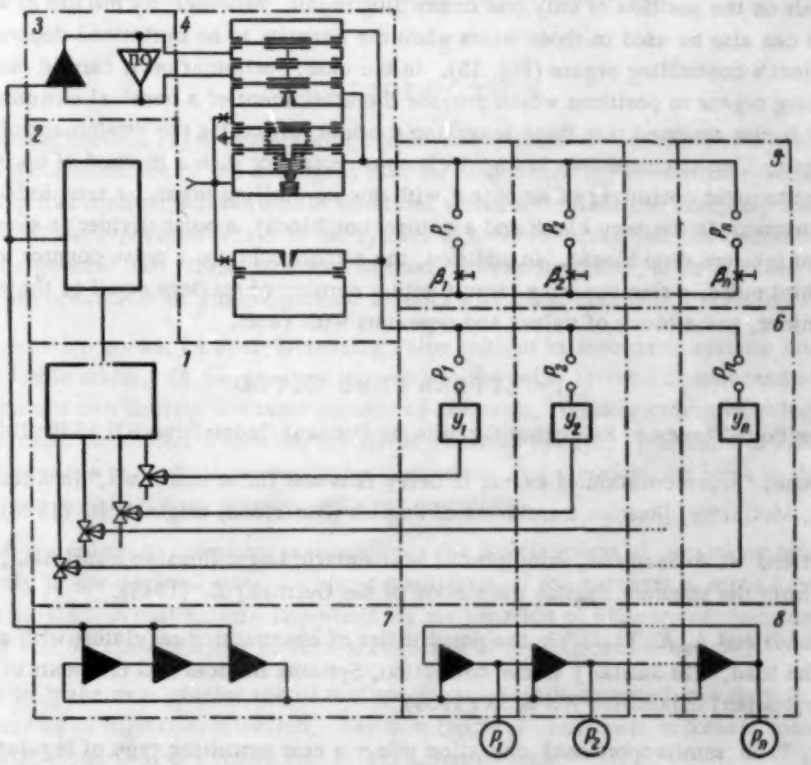

Fig. 15. Automatic optimizer of an object with several controlling organs. 1) Object with $\underline{n}$ controlling organs; 2) the automatic optimizer's memory block and comparison block; 3) a pulse divider; 4) integrator; 5) block of integrator wolves with vents; 6) block of power amplifiers with vents; 7) is the pulse counter; 8) ring counter circuit.

Some Applications of Digital Pneumatic Switching Devices

A Pneumatic Isodrome Regulator with Automatic Changeover. The ordinary controlling block of a unified system aggregate (USA) realizes a proportional-integral control law, and has two adjusting organs. One organ

establishes the regulator gain, the other the isodrome time constant. The tunings are maintained invariant during regulator operation. In many cases, for example, when the object's properties depend to a large degree on the load, and the device operates over a wide range of loads, difficulties arise in choosing a regulator tuning which would be equally correct for all modes of object operation. As was shown in [4], it is meaningful here to use a regulating block with an attachment to provide automatic variation of the regulator tuning when some external condition or another changes. This attachment (Fig. 13) consists of an interval indicator and a block of valves and vents. When the regulator operates with such an attachement, there is automatic variation of the regulator tuning whenever the object's mode of operation changes (as measured by pressure P at the input of the interval indicator) from one interval of values to another.

A Pneumatic Semiproportional Regulator. In industrial automata, the most widely used regulators are those which implement regulation laws (linear) which are constructed of some combination of proportional, integral and differential components. USA controlling blocks and lead blocks allow the realization of any of these laws. In [5], attention was turned to the advantages of using, in some cases, regulators which implement the (nonlinear) semiproportional control law suggested by V. Ferner. It is shown on Fig. 14, how by joining to a standard pneumatic USA controlling block an attachement containing two receiving pneumatic relays, one logical element (equivalence) and two vents, a proportional-integral block can be transformed to a semiproportional one.

An Automatic Pneumatic Optimizer of an Object with Several Controlling Organs. The existing pneumatic automatic optimizers for ordinary industrial use [6] are designed for operation on objects in which the quantity to be optimized depends on the position of only one controlling organ. However, by the use of a special attachment, this same optimizer can also be used in those cases when the quantity to be optimized depends on the position of several of the object's controlling organs (Fig. 15). In the case, optimization is carried out by successively moving the controlling organs to positions which provide the attainment of a nominal extremum of the quantity to be optimized. It is also assumed that three searching motions suffice for the attainment of such a position by each controlling organ. The circuit shown in Fig. 15 is constructed for such a method of optimizing. It includes the elements of an automatic optimizer of an object with one controlling organ: a sampled-data measurer of distance from an extremum (a memory block and a comparison block), a pulse divider (a stepping switch) and an integrator (a constant pressure drop block). In addition, the circuit contains a pulse counter which marks the appearance of every third pulse, a ring counting circuit with a number of sections equal to the number of controlling organs of the object, and a block of valves and repeaters with vents.

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PUNCHED-CARD METHOD OF SYNTHESIZING SEQUENTIAL SYSTEMS OF MULTIPOSITION RELAYS

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A punched-card method of synthesizing sequential systems of multiposition relays is presented. This method is valid for the synthesis both of autonomous and nonautonomous switching circuits, and is a generalization of the punched-card method of synthesizing sequential systems of two-position relays.

INTRODUCTION

A vector algebraic method [1] of synthesizing sequential systems of r-position relays has been presented, this method being a generalization of the analogous method of synthesizing sequential systems of two-position relays [2-4]. By using this method, one can obtain functions which formulate the necessary and sufficient conditions of operation of all the r-position relays in the system to be synthesized and the sequence of external stimuli to this system. In the general case, these functions are not Boolean functions, as in the case of systems of two-position relays, but are operations of a more general algebra which has been put forward by D. Webb *.

As digital automation grows, an ever-increasing value inheres in automatic systems constructed of relays with more than two stable states. Of the greatest interest are the relay systems constructed of three-position relays, since such systems can contain the least number of elements. Work is currently going on in the designing and building of automatic computers which use the ternary number system. This number system is the most economical, as concerns the equipment necessary for computers, and therefore the work in designing computers, operating in the ternary system, is very pressing and very promising.

Another pressing problem in computer technology is the development of computers using ten-position relays for computing directly in the decimal system without translation of the numbers to some other number system. The solution of this problem is particularly important for the creation of high-speed computers designed for financial computations and for computations in various branches of the national economy.

The tendency to make ever greater use of multiposition relays is observed, not only in computer technology, but also in other domains of digital automation. The development of discrete automata makes the problem of synthesizing sequential systems of multiposition relays an urgent one.

To solve this problem, one must use a new algebraic apparatus, namely, the algebra of multivalued logic and in particular, the algebra of Webb. The use of this method entails a large number of computations in the synthesis of sequential systems of multiposition relays. As the number of independent signals and relays in the system to be synthesized increases, the number of these computations grows very rapidly, much more rapidly than the number of analogous computations necessary for synthesizing seqeuential systems of two-position relays. Therefore, the requirement of mechanizing these computations arises.

This paper presents one of the simplest methods of mechanizing the vector algebraic method of synthesizing sequential systems of r-position relays, our method being based on the use of special punched cards. This method

* The Appendix to this paper contains a short sketch of Webb's algebra.

is a generalization of the punched-card method [6] which was initially developed for the synthesis of sequential systems of two-position relays. It is assumed that the reader is acquainted with work [6].

The Vector Algebraic Method of Synthesizing Sequential Systems of r-Position Relays

The vector-algebraic method of synthesizing sequential systems of r-position relays has already been described in [1], and will therefore not be presented in detail here. We shall present only such information regarding this method, which is necessary for the presentation of its implementation by means of special punched cards.

For brevity, we shall henceforth refer to a sequential system of r-position relays simply as a system. If all relays of the given system have identical delays, and if the magnitude of the delay for relay operate equals the magnitude of the delay for relay release, and also if all controlled and controlling parameters of the system change virtually instantaneously and synchronously, then the change of state of this system can be described by the equation

$$\mathbf{y}(t+\tau) = \mathbf{f}(\mathbf{x}(t), \mathbf{y}(t)), \tag{1}$$

where $\mathbf{f}$ is some single-valued vector function and τ is the delay time. If we assume that $\tau = 1$ (which, without any loss of generality, may always be done), then the previous equation can be replaced by the following:

$$\mathbf{y}(j+1) = \mathbf{f}(\mathbf{x}(j), \mathbf{y}(j)), \tag{2}$$

where $j = 0, 1, 2, \ldots$.

Equations (1) and (2) differ from the corresponding equations describing systems of two-position relays only in that the vectors $\mathbf{x}$ and $\mathbf{y}$ which enter into these equations are no longer binary, but are r-place vectors, where $r > 2$, and $\mathbf{f}$ denotes, not a Boolean function, but an arbitrary function of Webb's algebra.

The set of sequences $\{\mathbf{x}(j)\}$ and $\{\mathbf{y}(j)\}$ is called complete if, among the terms of these sequences, there are contained all r^m values $\mathbf{x}_\alpha$ of vector $\mathbf{x}$ and all r^n values $\mathbf{y}_\beta$ of vector $\mathbf{y}$,

$$\alpha = 0, 1, \ldots, r^m - 1; \quad \beta = 0, 1, \ldots, r^n - 1.$$

It follows, for the single-valuedness of function $\mathbf{f}$, that only such sets of sequences $\{\mathbf{x}(j)\}$ and $\{\mathbf{y}(j)\}$ are valid for the synthesis of the system described by Eq. (2), which satisfy condition (8) in work [6] as applied to r-place vectors.

The algorithm for synthesizing the system described by Eq. (2) is given by the formula

$$f(\mathbf{x}, \mathbf{y}) = \sum_{s=1}^{N}{}' \sum_{j=0}^{N_s - 1}{}' p_{\alpha_j(s)}(\mathbf{x})\, p_{\beta_j(s)}(\mathbf{y})\, \mathbf{y}^{(s)}(j+1), \tag{3}$$

where $\sum'$ is the symbol for the generalization of the Boolean sum, defined in Webb's algebra [5] by the formula 1.011, N is the number of pairs of sequences $\{\mathbf{x}(j)\}$ and $\{\mathbf{y}(j)\}$ in the given complete set of them, N_s is the number of states in the s'th sequence, and $p_\alpha(\mathbf{x})$ and $p_\beta(\mathbf{y})$ are generalized constituent units.

The generalized Boolean sum of two r-place variables u and v is defined by the formula

$$u \vee v = \max(u, v). \tag{4}$$

For binary variables, this operation coincides with the operation of Boolean addition

The generalized constituent units $p_\alpha(\mathbf{x})$ are defined by the following conditions:

$$p_\alpha(\mathbf{x}) = \begin{cases} 1, & \text{if} \quad \mathbf{x} = \mathbf{x}_\alpha \\ 0, & \text{if} \quad \mathbf{x} \neq \mathbf{x}_\alpha. \end{cases} \tag{5}$$

The generalized constituents $p_\beta(\mathbf{y})$ are defined analogously.

The values of $\mathbf{x}_\alpha$ and $\mathbf{y}_\beta$ of the variable r-place vectors $\mathbf{x}$ and $\mathbf{y}$ will be numbered by integers α and β by the same rules as in work [1], i.e., by the rules expressed by the formulas

$$\alpha = \sum_{i=1}^{m} a_i r^{i-1}, \qquad \beta = \sum_{k=1}^{n} b_k r^{k-1}, \tag{6}$$

where a_i and b_k are the i'th and k'th components of the vectors

$$\mathbf{x}_\alpha = [a_1, \ldots, a_i \ldots, a_m], \qquad \mathbf{y}_\beta = [b_1, \ldots, b_k, \ldots, b_n], \tag{7}$$

respectively.

The generalized constituents $p_\alpha(\mathbf{x})$ and $p_\beta(\mathbf{y})$ are expressed in terms of the components of the vectors $\mathbf{x}_\alpha$, $\mathbf{x}$, $\mathbf{y}_\beta$ and $\mathbf{y}$ by the following formulas:

$$p_\alpha(\mathbf{x}) = \prod_{i=1}^{m} \delta_{x_i, a_i}, \qquad p_\beta(\mathbf{y}) = \prod_{k=1}^{n} \delta_{y_k, b_k}, \tag{8}$$

where $\delta_{xi,ai}$ and $\delta_{yk,bk}$ are the well-known Kronecker deltas, i.e., functions defined by the conditions

$$\delta_{z,c} = \begin{cases} 1, & \text{if} \quad z = c, \\ 0, & \text{if} \quad z \neq c. \end{cases} \tag{9}$$

For an autonomous system, i.e. a system described by the equation

$$\mathbf{y}(j+1) = f(\mathbf{y}(j))$$

all the formulas given above are simplified in an obvious manner.

Punched Cards for Synthesizing Autonomous Systems

The vector algebraic method described above for the synthesis of autonomous systems of any r-position relays is a natural generalization of the similar method for the particular case of two-position relays, and therefore, the punched cards intended to mechanize this method in the general case when ≥ 2 can be made very little different from the punched cards for the synthesis of autonomous systems of two-position relays. In the general case, as in the previously described [6] particular case of two-position relays, for each value $\mathbf{y}$ of a variable n-dimensional vector $\mathbf{y}$ there is prepared an individual punched card, "punched card β" and, consequently, ther must always be prepared r^n different "punched cards β".

Since it is almost impossible, with manual distribution of cards and visual reading of their numbers, to function with more than a thousand cards, the maximum number n, for a given r, may be defined by the inequality $r^n \leq 1000$, from which it follows, in particular, than = 6 for r = 3 and n = 3 for r = 10. In fact, $3^6 = 729 \leq 1000 < 3^7 = 2187$ and $10^3 \leq 1000 < 10^4$, i.e., the set of "β" cards for the synthesis of autonomous sytems must contain at most 729 punched cards in the case of three-position relays, and 1000 cards in the case of ten-position relays. Since both the number of cards in the set and the very construction of the cards must depend on r, we shall henceforth denote the punched card corresponding to the β'th value $\mathbf{y}_\beta$ of the variable r-place vector $\mathbf{y}$ by the symbol $\beta(r)$, and shall call this punched card "punched card $\beta(r)$". We shall, in correspondence with this condition, call the punched cards described in work [6] for the synthesis of autonomous systems of two-position relays "punched cards β (2)". The sole difference, in the general case being considered here, is that each component b_k

of vector $\mathbf{y} = [b_1, \ldots, b_k, \ldots, b_n]$ will assume, not one of two values: 0 or 1, but one of r values: $0, 1, \ldots, r-1$. Thus, punched cards $\beta(r)$ can be prepared so that they differ from previously described [6] punched cards $\beta(2)$ only for $b_k = 2, 3, \ldots, r-1$.

By adhering to the "principle of least difference" of punched card $\beta(r)$ from punched card $\beta(2)$, we set aside the left half of punched card $\beta(r)$, as was previously done for cards $\beta(2)$, for the perforation of the value of component b_k of vector y_β, where the hole (perforation) corresponding to the value of the k'th component b_k is found in k'th row (counting down from above) of punched card $\beta(r)$. In the case when $b_k = 0$, there will be no punch in the k'th row. The value $b_k = 1$, as in the case $\beta(2)$, corresponds to a rectangular punch at the edge of the k'th row, which, certainly, must not extend the entire width of the left half of punched card $\beta(r)$, since it is necessary to leave space for the perforations corresponding to the values of $b_k = 2, 3, \ldots, r-1$. For these values of b_k, there is made a single punch in the k'th row of cards $\beta(r)$ in the column corresponding to the given value of b_k, i.e., the punch is made at the intersection of the k'th row of punched card $\beta(r)$ with the second column if $b_k = 2$, or with the third column if $b_k = 3$ or, finally, with the $(r-1)$'st column if $b_k = r-1$. The rectangular hole, or punch, must have dimensions which allow one to see through the aperture all three digits of number β, of the punched card on which the given punched card is lying. By bearing in mind that the distance between punches equals the width of one digit, we obtain the fact that the width of the left half of a punched card must be not less than $4(r-1)$ times larger than the width of one digit. The right half of punched card $\beta(r)$ must have exactly the same width, since on them must be placed $r-1$ columns of three-place decimal numbers, each of which is the serial number of the given punched card $\beta(r)$.

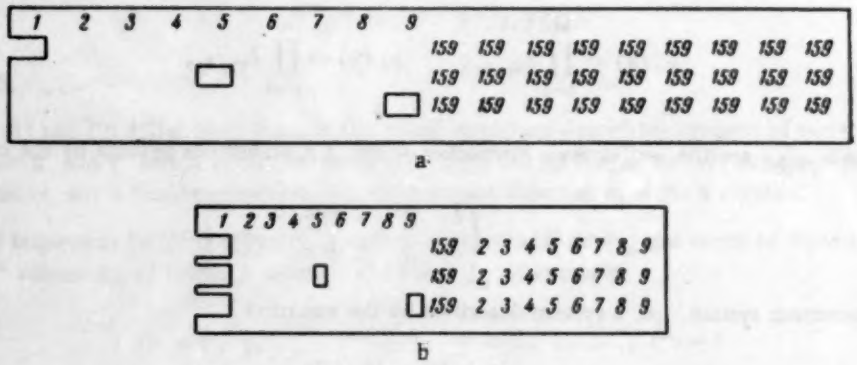

Fig. 1

Figure 1, a shows punched card $\beta(10) = 159$, prepared exactly in accordance with the method just described.

The width of the punched cards can be decreased by a factor of one and a half to two, if their construction is changed somewhat. In order to obtain such a decrease in punched card width, it is necessary to do the following:

1) To make the same rectangular punches as in the punched cards of the former construction not only for $b_k = 1$, but also for $b_k = 2, \ldots, r-1$.

2) For $b_k = 2, \ldots, r-1$, in the left half of the punched card, to punch an opening of one digit width at the intersection of the k'th row and the corresponding column.

3) In the right portion of punched card, to leave the ordinal number β of this punched card in just the first (left-most) column, and fill the remaining columns, II, III and finally $(r-1)$, with the corresponding numbers $2, 3, \ldots, (r-1)$.

Figure 1,b shows punched card $\beta(10) = 159$, prepared by this method.

Punched cards $\beta(3)$ prepared by the method just described are approximately half the size of those prepared by the previous method. Figure 2 shows the punched cards for $\beta(3)$:

$$0(3),\ 1(3),\ 2(3),\ 3(3),\ 4(3), \ldots, 25(3),\ 26(3),$$

prepared by the second of the methods described above.

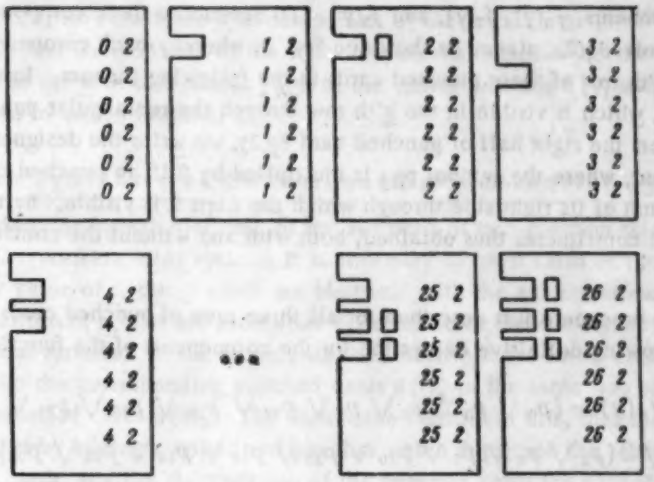

Fig. 2

The $\beta(3)$ punched cards omitted from Fig. 2 are easily reproduced by using a table which gives the first 27 values y_β of the six-place ternary vector y.

Punched cards $\beta(r)$ are used for the synthesis of autonomous systems of r-position relays analogously to the way punched cards $\beta(2)$ are used for the synthesis of autonomous two-position relay systems. The punched -card method of synthesizing the latter was described in detail in [6]. Therefore, it is not necessary to present the method of using punched cards $\beta(r)$ in the general form. We limit ourselves her to the consideration of one example of the use of punched cards $\beta(3)$ for the synthesis of an autonomous three-position relay system.

Example. Let it be required to construct an autonomous system of three-position relays y_1, y_2 and y_3 such that they operate in the following sequence, $(y_0, y_1, y_2, \ldots, y_{25}, y_{26})$, i.e., so that all the 27 possible states of the system of these three-position relays repeat periodically with period 27, and follow one another in the natural sequence given in the table. In other words, let it be required, for a given periodic sequence of values $\{y_{\beta_j(3)}\}$ of the three-place ternary vector y which describes the state of the desired system, to find the function f, which satisfies Eq. (2a). For this, we take punched cards $\beta(3)$, represented on Fig. 2, and place them one after the other in the sequence

$$0(3),\ 1(3),\ 2(3),\ 3(3),\ \ldots,\ \beta_j(3),\ \beta_{j+1}(3),\ \ldots,\ 25(3),\ 26(3),\ 0(3)$$

such that the right half of each preceding card $\beta_j(3)$ is covered by the left (perforated) half of the succeeding card $\beta_{j+1}(3)$. Figure 3 shows the initial and final fragments of this placement of punched cards $\beta_j(3)$. The reader can easily regenerate the omitted middle portion by use of the table since, for the case considered, we have $\beta_j = j$ for each value of $j(j = 0, 1, \ldots, 26)$.

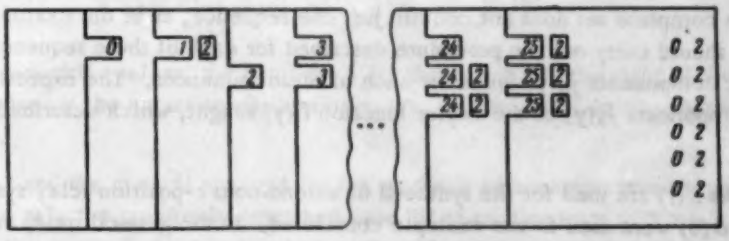

Fig. 3.

Punched cards $\beta(3)$ shown in Figs. 2 and 3 are useful for the synthesis of autonomous system which do not contain more than 6 relays $V_1, V_2, \ldots, V_6$. In particular, they are valid for the synthesis of the autonomous system of our example, in which the system contained only three relays. In this case, the desired function $f(y)$

contains only three components, $f_1(y)$, $f_2(y)$, and $f_3(y)$. To determine these components, we use only the first three rows of punched cards $\beta_j(3)$, placed as shown on Fig. 3, whereby each component $f_k(y)$ is determined from the corresponding $\underline{k}$'th row of these punched cards in the following manner. Instead of the serial number β_j of punched card $\beta_j(3)$, which is visible in the $\underline{k}$'th row through the rectangular punch in the left half of punched card $\beta_{j+1}(3)$ which covers the right half of punched card $\beta_j(3)$, we write the designation p_{β_j} of the corresponding generalized constituent, where the symbol p_{β_j} is multiplied by 2 if, in punched card $\beta_{j+1}(3)$, there is a punch in the second column of its right side through which the digit 2 is visible. By taking the genralized Boolean sum of all the generalized constituents thus obtained, both with and without the coefficient 2, we obtain the component $f_k(y)$.

By carrying out the procedure just described for all three rows of punched cards $\beta_j(3)$, placed as shown on Fig. 3, we obtain the following definitive expression for the components of the function $f(y)$ sought:

$$f_1(y) = (p_0 \vee p_3 \vee p_6 \vee p_9 \vee p_{12} \vee p_{15} \vee p_{18} \vee p_{21} \vee p_{24}) \vee$$
$$\vee 2(p_1 \vee p_4 \vee p_7 \vee p_{10} \vee p_{13} \vee p_{16} \vee p_{19} \vee p_{22} \vee p_{25}),$$
$$f_2(y) = (p_2 \vee p_3 \vee p_4 \vee p_{11} \vee p_{12} \vee p_{13} \vee p_{20} \vee p_{21} \vee p_{22}) \vee$$
$$\vee 2(p_5 \vee p_6 \vee p_7 \vee p_{14} \vee p_{15} \vee p_{16} \vee p_{23} \vee p_{24} \vee p_{25}),$$
$$f_3(y) = (p_8 \vee p_9 \vee p_{10} \vee p_{11} \vee p_{12} \vee p_{13} \vee p_{14} \vee p_{15} \vee p_{16}) \vee$$
$$\vee 2(p_{17} \vee p_{18} \vee p_{19} \vee p_{20} \vee p_{21} \vee p_{22} \vee p_{23} \vee p_{24} \vee p_{25}).$$

The components $f_1(y)$, $f_2(y)$ and $f_3(y)$ are the necessary and sufficient conditions for the operation of relays Y_1, Y_2, and Y_3, respectively. It is easily verified that the expressions found for these components guarantee the operation of the system of these relays exactly as in the cyclical sequence given above.

β	$b_{1,}$	$b_{2,}$	$b_{3,}$	...	b_{n}	β	$b_{1,}$	$b_{2,}$	$b_{3,}$	...	$b_{n,}$
0	0	0	0	...	0	14	2	1	1	...	0
1	1	0	0	...	0	15	0	2	1	...	0
2	2	0	0	...	0	16	1	2	1	...	0
3	0	1	0	...	0	17	2	2	1	...	0
4	1	1	0	...	0	18	0	0	2	...	0
5	2	1	0	...	0	19	1	0	2	...	0
6	0	2	0	...	0	20	2	0	2	...	0
7	1	2	0	...	0	21	0	1	2	....	0
8	2	2	0	...	0	22	1	1	2	...	0
9	0	0	1	...	0	23	2	1	2	...	0
10	1	0	1	...	0	24	0	2	2	...	0
11	2	0	1	...	0	25	1	2	2	...	0
12	0	1	1	...	0	26	2	2	2	...	0
13	1	1	1	...	0						

In case the given complete set does not contain just one sequence, as in the example just considered, but several sequences, one should carry out the procedure described for each of these sequences, and then carry out Boolean addition of the components $f_k(y)$ found for each of these sequences. The expressions thus obtained would be, in this case, the components $f_k(y)$ of the vector function $f(y)$ sought, which described the autonomous system to be synthesized.

The punched cards $\beta(r)$ are used for the synthesis of autonomous r-position relay systems analogously to the way punched cards $\beta(3)$ were used in the example considered. In the general case, instead of the serial number β_j of punched card $\beta_j(r)$, which is visible through marginal rectangular punch in the $\underline{k}$'th row of punched card $\beta_{j+1}(r)$ covering the right half of punched card $\beta_j(r)$, one writes the symbol p_{β_j} of the corresponding generalized consituent $p_{\beta_j}(y)$, where this symbol is multiplied by the number which is visible through the punch of the same punched card $\beta_{j+1}(r)$ if, to be sure, such a punch exists. If no such punch exists, then only the symbol p_{β_j} is written. If, in the $\underline{k}$'th row of punched card $\beta_{j+1}(r)$, there is neither a marginal punch nor a hole, then, naturally, the symbol p_{β_j} will be lacking for this row.

By taking the generalized Boolean sum of all the generalized constituents thus obtained, both with and without the coefficients 2, 3, . . ., r, we obtain the k'th component of the vector function $f^{(\rho)}(y)$ defined by the ρ'th given sequence. The generalized Boolean sum of the k'th components found for all the sequences entering into the given complete set will be the k'th component $f_k(y)$ of the vector function f (y) sought, describing the autonomous r-position relay system to be synthesized.

Punched Cards for the Synthesis of Nonautonomous Sequential Systems

As for the previously described particular case of the synthesis of two-position relays [6], for the synthesis of nonautonomous sequential r-position relay systems it is necessary to have cards of two forms – α cards and $\beta(r)$ punched cards. For any value of r, the α cards are identical with the analogous cards as described in [6]. Cards $\beta(r)$ are different for different r, and are somewhat different from the corresponding punched cards intended for the synthesis of autonomous systems. In the sequel, we shall denote these latter cards by the symbol $\beta(r)_a$. Punched cards $\beta(r)$ differ from the corresponding punched cards $\beta(r)_a$ in the same way as punched cards $\beta(2)$ differ from the corresponding punched cards $\beta(2)_a$. The difference consists in this, that the perforated portion of punched cards $\beta(r)$ is extended by adding a column of punches which duplicate the rectangular apertures on the left margins of the punched cards, and the right portion of the punched cards are extended by an empty field along their right margins. The width of this empty field on the $\beta(r)$ punched cards must equal the width of the α cards, since it is intended precisely for the placement of the α cards in the implementation of the synthesis of nonautonomous sequential systems.

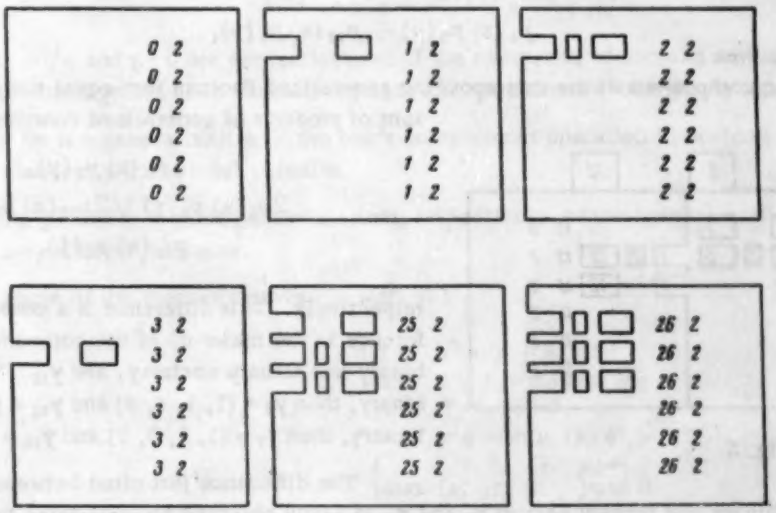

Fig. 4

Figure 4 shows the punched cards

$$\beta(3) = 0(3),\ 1(3),\ 2(3),\ 3(3), \ldots,\ 25(3),\ 26(3).$$

Just as in the case of two-position relays, the numbers β (in the column to the right of the column of duplicate apertures) on cards $\beta(r)$ denote the natural constitutents $p_\beta(y)$, and the serial number borne by the card denotes the α-constituent $p_\alpha(x)$.

Punched cards $\beta(r)$ and the α cards are used for the synthesis of nonautonomous r-position relay systems in exactly the same manner as in the previously described case [6] of such systems with r = 2. Therefore, we will not describe again here this method of using punched cards $\beta(r)$ and the α cards in the general case of an arbitrary r. We will limit ourselves to a discussion of Fig. 5, which shows the set of punched cards

$$\beta_j(3) = 3,\ \alpha_j = 5,\ \beta_{j+1}(3) = 7,\ \alpha_{j+1} = 12,\ \beta_{j+2}(3) = 15,$$

which is the fragment of the distribution of punched cards $\beta_j(3)$ and α cards in the sequence:

$$\beta_0(3),\ \alpha_0,\ \beta_1(3),\ \alpha_1, \ldots,\ \beta_j(3),\ \alpha_j,\ \beta_{j+1},(3),\ \alpha_{j+1},\ \beta_{j+2}(3), \ldots,$$

corresponding to the given (for synthesis) sequence of values of the ternary vectors $\mathbf{y}(j)$, $\mathbf{x}(j)$:

$$\mathbf{y}_{\beta_0},\ \mathbf{x}_{\alpha_0},\ \mathbf{y}_{\beta_1},\ \mathbf{x}_{\alpha_1}, \ldots,\ \mathbf{y}_{\beta_j},\ \mathbf{x}_{\alpha_j},\ \mathbf{y}_{\beta_{j+1}},\ \mathbf{x}_{\alpha_{j+1}},\ \mathbf{y}_{\beta_{j+2}}, \ldots$$

By reading, through the apertures of the punched cards shown on Fig. 5, the numbers $\beta_j(3)$ and α_j, we obtain the following fragment of the generalized Boolean sum:

$$\vee\ p_5(\mathbf{x})\, p_3(\mathbf{y})\mathbf{y}_7 \vee p_{12}(\mathbf{x})\, p_7(\mathbf{y})\, \mathbf{y}_{15} \vee .$$

This expression is analogous to the fragment of the Boolean sum

$$+\ p_5(\mathbf{x})\, p_3(\mathbf{y})\, \mathbf{y}_7 + p_{12}(\mathbf{x})\, p_7(\mathbf{y})\, \mathbf{y}_{15} +,$$

obtained from Fig. 9 of work [6], and differs from it only in the set of components. The first three components of the Boolean sum just given (1) are identical, and equal the Boolean sum of the products of constituent units

$$p_5(\mathbf{x})\, p_3(\mathbf{y}) + p_{12}(\mathbf{x})\, p_7(\mathbf{y}),$$

while the first three components of the corresponding generalized Boolean sum equal the following generalized sum of products of generalized constituent units

$$p_5(\mathbf{x})\, p_3(\mathbf{y}),$$
$$2p_5(\mathbf{x})\, p_3(\mathbf{y}) \vee 2p_{12}(\mathbf{x})\, p_7(\mathbf{y}),$$
$$p_{12}(\mathbf{x})\, p_7(\mathbf{y}),$$

respectively. This difference is a consequence of the difference in the make-up of the components of the single-place binary and ternary vectors $\mathbf{y}_7$ and $\mathbf{y}_{15}$. If these vectors are binary, then $\mathbf{y}_7 = [1, 1, 1, 0]$ and $\mathbf{y}_{15} = [1, 1, 1, 1]$. If they are ternary, then $\mathbf{y}_7 = [1, 1, 0, 0]$ and $\mathbf{y}_{15} = [0, 2, 1, 0]$.

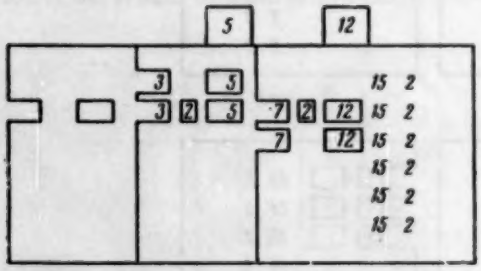

Fig. 5

The difference just cited between the make-ups of the components of the binary and ternary vectors $\mathbf{y}_7$ and $\mathbf{y}_{15}$ is easily observed by comparing the perforations in the punched cards 7(2), 15(2) and 7(3), 15(3), represented on Fig. 9 of work [6] and on Fig. 5 of the present paper.

We note in conclusion that the $\beta(r)$ punched cards can also be used instead of the $\beta(r)_a$ punched cards in synthesizing autonomous systems. With this, of course, the α cards are not necessary. The duplicating apertures and the empty fields on punched cards $\beta(r)$ will not then be used, and should be given no further notice.

With manual positioning of punched cards $\beta(r)$ and of the α cards, and with visual reading of the digits through the apertures in punched cards $\beta(r)$, the punched card method described can be practicably used for the synthesis of relay systems with a total number of independent and dependent variables which does not exceed six, for $r = 3$, and three for $r = 10$. With mechanization of the punched-card method by means of punched card (EAM) machines, the total number of relays and independent variables can be doubled and, seemingly, even tripled.

APPENDIX

A Short Sketch of the Algebra of D. Webb

In a paper [5] published in 1936, D. Webb described an algebra, suggested by him, of an n-valued logic, this being a generalization of the two-valued Boolean algebra generally used as the classical propositional calculus.

Using the author's somewhat simplified notation, we will here denote by the letters p, q and r variables which can assume the values 0, 1, . . ., n, but we shall retain the numeration of the formulas used in the paper of Webb which was cited earlier.

The fundamental operation in Webbian algebra is the operation defined by the equation:

$$1.01 \quad p/q = 1 + \max(p, q),$$

where the sign + (plus) denotes addition modulo n. In what follows, the signs + (plus) and − (minus) will always be used as the signs for adition and subtraction modulo n, so that we shall not bother writing the usual symbol (mod n) after each equation in which these signs appear.

The operation p/q, defined by Equation 1.01, is a generalization of the well-known Sheffer stroke operation. All other operations of Webbian algebra are defined in terms of this operation:

$$1.02 \quad p^0 = p, \quad p^{i+1} = p^i | p^i \quad (i = 0, 1, \ldots, n-1)$$

$$1.03 \quad p \vee q = (p|q)^{n-1}$$

$$1.04 \quad Np = p^{2n-2p-1}$$

$$1.05 \quad p \cdot q = N(Np \vee Nq)$$

$$1.06 \quad p \supset q = (p \cdot q)^{n-1-p}$$

$$1.07 \quad p \equiv q = (p \supset q) \cdot (q \supset p)$$

$$1.08 \quad p \times q = p^{n-1} \vee q^{n-1}.$$

The operation $p \vee q$ and $p \cdot q$ are generalizations of the operations of Boolean addition and multiplication respectively. In Webbian algebra, as in Boolean algebra, one sometimes writes simply pq, instead of $p \cdot q$.

The operation Np is a generalization of the one's complement operation in Boolean algebra, or the operation of negation in classical propositional calculus.

The operation $p \supset q$ and $p \equiv q$ are generalizations, respectively, of the operations of implication and equivalence in classical propositional calculus.

We now give some of the theorems proved in work [5]:

$$1.1 \quad p^h = p + h,$$

$$1.2 \quad p \vee q = \max(p, q),$$

$$1.3 \quad Np = n - 1 - p$$

$$1.4 \quad p \cdot q = \min(p, q),$$

$$1.15 \quad p \times q = \begin{cases} 0, & \text{if } pq = 0 \\ \max(p, q), & \text{if } pq \neq 0 \end{cases}$$

Operation $p \times q$ is distributive with respect to the operation of generalized Boolean addition, i.e., the following theorem 1.25 is valid

$$p \times (q \vee r) = (p \times q) \vee (p \times r).$$

By virtue of the commutativity and associativity (proven in [5]) of the operations $p \vee q$, $p \cdot q$, we can agree to write the expressions

$$p \vee (q \vee r), \quad p \cdot (q \cdot r) \text{ и } p \times (q \times r)$$

without parentheses, i.e., to introduce the definitions:

$$1.09 \quad \begin{cases} p \vee q \vee r = p \vee (q \vee r), \\ p \cdot q \cdot r = p \cdot (q \cdot r), \\ p \times q \times r = p \times (q \times r). \end{cases}$$

We introduce, by definitions, the following abbreviated notation:

1.010 $X_i = x^0 x^1 \ldots x^{i-1} x^{i+1} \ldots x^{n-1}$

1.011 $\sum_{i=0}^{n-1} x_i = x_0 \vee x_1 \vee \ldots \vee x_i \vee \ldots \vee x_{n-1}$

and analogously for $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \cdots \sum_{k=0}^{n-1} x_{i, j, \ldots, k}$, where x_i and $x_{i, j, \ldots, k}$ are any elementary propositons of n-valued logic.

The symbol $\sum$, introduced by definition 1.011 is, for n = 2, the symbol for the Boolean sum (or disjunction, to use the terminology of the classical propositional calculus). In the general case, when n is arbitrary, this symbol will be called the symbol for the generalized Boolean sum.

According to the definition 1.010, for n = 2, i.e., in Boolean algebra, there exist the two functions $X_0 = x^1 = \overline{x}$, $X_1 = x^0 = x$, where $\overline{x}$ is the one's complement of x.

From theorems 1.27 – 1.33, proven in work [5] follows the formula $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \cdots \sum_{k=0}^{n-1} (X_i \times Y_j \times Z_k) = 1$, on the basis of which the functions $X_i \times Y_j \times Z_k$ can be called generalized constituent units.

The following theorem was proven in [5]:

$$F(x, y, \ldots, z) \equiv \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \cdots \sum_{k=0}^{n-1} \{F(i, j, \ldots, k) \times X_{n-i} \times Y_{n-j} \times Z_{n-k}\},$$

which asserts than any function F(x, y, . . ., z) of n-valued logic can be expanded in terms of generalized constituent units or, as expressed by Webb, can be given in the normal form:

$$F(i, j, \ldots, k) \times X_{n-i} \times Y_{n-j} \times \ldots \times Z_{n-k}.$$

The normal form of a function F(x, y, . . ., z) of n-valued logic is a generalization of the complete normal disjunctive form of a proposition in the classical propositional calculus.

A number of purely logical questions were also considered in work [5], in particular, the relationship of Webbian n-valued logic to other systems of multivalued logic and to the logical scheme presented in the Principia Mathematica.

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SHIFT REGISTERS WITH LOGICAL FEEDBACK AND THEIR USE AS COUNTING AND CODING DEVICES

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Methods are considered for increasing the efficiency of counting and coding devices based on shift registers. The distinguishing feature of such devices is that the shift register is closed into a ring, whereby the closing link is a logical circuit, whose state is a function of the code introduced into the register. Methods are presented for constructing logical feedback circuits in order to obtain counting devices or coding devices of arbitrary capacity, and methods of reading out information are also given.

In various devices for remote control and computing, use is frequently made of pulse counting circuits built of binary flip-flop circuits in the role of code combination transducers. However, in many cases, serious difficulties arise in the use of such schemes. This springs from the inevitable delay when signals are transmitted from low-order bits to higher-order ones, these delays being cumulative and, depending on the number of bits, of a greater or a smaller total value. As a result, false combinations may occur during the transient responses. For example, the transition of an ordinary binary counter from the state 0111 to the next state, 1000, is accompanied by a series of false states (shown in parentheses): 0111 → (0110 → 0100 → 0000) → 1000. The most noticeable are the appearances of false code combinations in comparatively slow devices, for example, in relay-contact ones, although it is obvious that the relative lag, expressed as the number of cycles of lag to the high-order bits as compared to the low-order bits, will be approximately identical for both slow and fast counters, if each of them operates near the upper threshold of its speed of action. The lag of the high-order bits with respect to the low-order bits can reach 5 to 10 cycles. If the counter circuit is being used as a delay line, as occurs, for example, in dynamic flip-flops, then the discrepancy can attain an even greater value.

In order to eliminate the cause of false combination formation, it is necessary that the transition to each successive fixed state be occasioned by the operation of the flip-flop for one bit only. This route is a well-known one today. It is based on the use of special counters which operate in correspondence with the sequences of a Gray code. However, this is not the only method for eliminating false intermediate combinations. The nonsimultaneity of flip-flop operations, which is the cause of false code combination appearance, can be avoided if one makes the transition from a serial flip-flop circuit to a parallel one, i.e. one in which the pulses being counted are applied to all bits simultaneously.

A shift register can serve as the basis for the construction of such a counter circuit. Simultaneous action of all its cells (bits) is provided by the shift bus in which the pulses to be counted are introduced. The elements entering into a shift register may be quite diverse: electronic, semiconducting, magnetic, etc. The method presented below for constructing computing and coding circuits is applicable to any case. However, the most technological and economic advantage is gained by the use of shift registers of magnetic elements with rectangular hysteresis loops. A condition for this is that the usual disadvantage of such elements – the necessity of special means for reading out information – is completely excluded in the present case, this necessity also being lacking when dynamic triggers (flip-flops) are used. With this, particularly for low counting speeds, there

there is a sharp drop in power requirements, since magnetic registers ordinarily do not have internal power supplies and require power only during the transient responses. As a result, the circuits constructed in accordance with the method described below are some two to three times as simple and economical as flip-flop circuits of equivalent capacity. An additional superiority of shift-register circuits is the absence of errors in dynamic modes of operation.

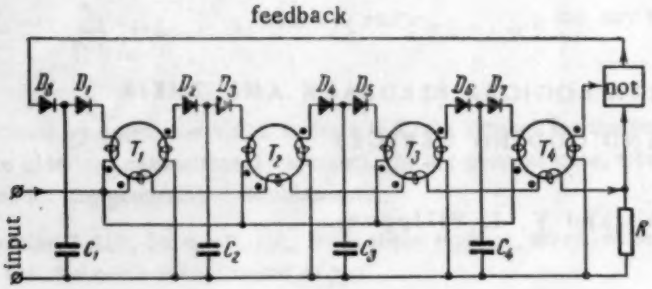

Fig. 1

To convince ourselves that, with a shift register closed in a ring, one can realize more than n discrete states (where n is the number of register cells), we consider the circuit of Fig. 1. It contains four ferrite cores, T_1, T_2, T_3 and T_4, joining to a common circuit by means of diodes $D_1 - D_8$ and capacitors C_1, C_2, C_3 and C_4. The distinguishing feature of the circuit is the use of the logical "not" element in the feedback path, whose presence permits a doubling of the capacity of any analogous ferrite circuit. In the given case, the capacity is increased from four to eight, in a decade counter the circuit would permit counting up to 20, etc.

Let us follow through the operation of the circuit. At the initial moment, let all cores be so magnetized that a pulse applied to the input does not induce the reversal of their magnetic states. This pulse is divided on resistor R and passed through the "not" circuit, charging capacitor C_1. Until the termination of the pulse C_1 cannot be discharged, since diode D_1 is cut off by the voltage of the input pulse which acts on the diode's cathode via the input winding of T_1. After the termination of the pulse, the cut-off voltage falls and condenser C_1, in discharging through D_1, the input winding of T_1 and the internal impedance of the pulse source, reverses the magnetic polarity of core T_1. A second pulse to be counted returns core T_1 to its previous state, whereby C_2 is charged. Simultaneously, the second pulse passes through the "not" circuit and again charges condenser C_1. When this pulse terminates, condenser C_1 and C_2 discharge through the windings of T_2 and T_1, and reverse the magnetic polarities of these cores. The character of circuit operation does not change until all the cores have had their magnetic polarities reversed, which occurs at the termination of the fourth pulse to be counted. The operation cycles so far considered can be conventionally expressed by the following table:

0000 – initial state
1000 – after application of the first pulse
1100 – after application of the second pulse
1110 – after application of the third pulse
1111 – after application of the fourth pulse.

Here, a zero denotes a nonexcited core state, and a one denotes a state of reversed magnetic polarity.

The character of circuit operation now changes: the pulses divided by resistor R will not be sent to the feedback path, since there will simultaneously arise pulses at the output winding of T_4 which will cut off the "not" circuit. As a result, condenser C_1 will be discharged at the fifth cycle, and the circuit will go to the state 0111. Thus, circuit operation continues in accordance with the following table:

0111 – fifth cycle
0011 – sixth cycle
0001 – seventh cycle
0000 – eight cycle (the initial state).

Thereafter, circuit operation repeats the previous sequence.

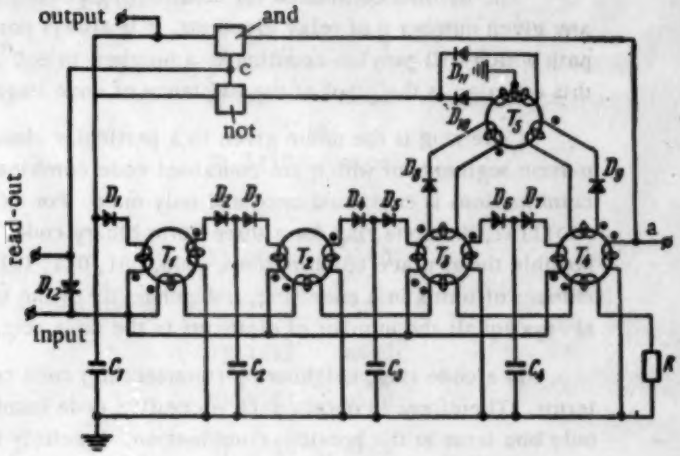

Fig. 2

This circuit will cary out counting up to 8. A circuit of n cells will count up to 2n.

The logical function in the feedback circuit essentially affects the counting factor of the circuit. Figure 2 shows the same counter as in Fig. 1, but with a different feedback circuit. Here, a core with a rectangular hysteresis loop is used in the feedback path. Pulses applied from core T_3 take it to the "1" state. Pulses applied from T_4 return T_5 to the "0" state. If pulses from T_3 and T_4 appear simultaneously, core T_5 does not change its state, since the corresponding magnetization polarity reversing windings act in opposition. Reversal of magnetization polarity is accompanied by a voltage on its output winding. This winding, in conjunction with diodes D_{10} and D_{11}, forms a two-half-period rectifier circuit at whose output appear pulses of positive polarity, independently of the direction in which the magnetic polarity of T_5 was reversed.

Let us consider circuit operation, with the assumption that, in the initial state, the first core is in the state "1" and the others are in state "0". The following table gives the sequence of circuit states:

1) 1000	6) 0110	11) 1110
2) 0100	7) 1011	12) 1111
3) 0010	8) 0101	13) 0111
4) 1001	9) 1010	14) 0011
5) 1100	10) 1101	15) 0001
		1) 1000 etc.

The first three states require no explanation. When the fourth pulse appears, core T_3 is returned to zero, thus occasioning the reversal of magnetic polarity of T_5 and charging of condensers C_4 and C_1 (via T_5). At the termination of the fourth cycle, the cores are excited in the combination 1001. The fifth pulse returns cores T_1 and T_4 to the "zero" state, with which core T_5 in the feedback path is also returned to the zero state. This leads to the charging of C_1 and C_2, which provides the appearance of the combination 1100, etc.

Transmission of a one to the first cell occurs if one of the cores T_3 and T_4 is in state "1". No transmission of ones occurs if both these cores are in state "0" or in state "1". A necessary condition for the transmission of units is the alternate action on T_5 by T_3 and T_4. It can be verified from the table that this condition is satisfied. In fact, by writing the individual states of T_3 and T_4, we get

T_3 0010011010111110

T_4 0001001101011111

The underlined ones transmit information to the feedback path. It is easily seen that they follow alternately in the upper and lower lines. Otherwise, creation of a feedback path with a core with a rectangular hysteresis loop would be impossible.

The feedback circuit shown in Fig. 2 is sufficiently universal, and can be used to construct code counters of arbitrary capacity. For example, a circuit of seven binary cells with the same feedback circuit provides counting up to 127.

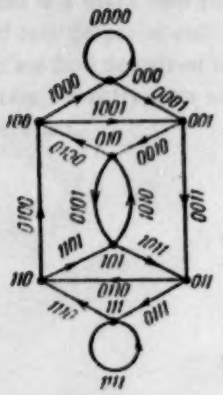

Fig. 3

The method considered for constructing counting circuits is general, i.e., for any given number n of relay elements, it is always possible to construct a feedback path which will provide counting to a number $m \leq 2^n$. The theoretical basis of this assertion is the proof of the existence of code rings [1].

Code ring is the name given to a particular closed sequence of symbols, in the n-term segments of which are contained code combinations, where each of these combinations is contained once and only once. For example, the closed sequence 00011101, the code ring for a three-term binary code, contains in its segments all possible three-place combinations: 000, 001, 011, 111, 110, 101, 010 and 100. The number of terms in a code ring, independently of the number of terms in each code, always equals the number of elements in the code ring.

In a code ring, neighboring (intersecting) code combinations have n-1 common terms. Therefore, to obtain each successive code combination, it suffices to replace only one term in the previous combination. Precisely this operation is implemented by a register with a shift of combinations by one step: n−1 terms are retained (shifted to more-significant orders) and, in place of the disappearing right-most term, the new term of the next combination of the code ring appears on the left by means of the feedback path. Since the sequence of symbols in a code ring is always known in advance, one can always determine the previous or following terms of the ring for any n-term combination. It follows from this that the logical circuit of the feedback path can be constructed by realizing a Boolean function which is given in the following form: the function's arguments are all the possible n-term code ring segments, and the values of the function are the code ring symbols which are preceded by the segment which is the current argument.

Therefore, the method of setting up the logical feedback path for a counting circuit reduces to the following.

1. Choice of the code ring corresponding to the conditions of the problem. The length of the code ring (the number of its terms) must equal the counting coefficient of the circuit to be developed (i.e., the number up to which it is supposed to count). A method for constructing such a ring is presented in [1]. Synthesizing code rings from the graphic representation of the set of code rings, as suggested in [2], can be recommended. Figure 3 shows such a set for four-term rings. To construct a code ring, one starts from some node and completes a circuit in the direction given by the arrows, returning to the original node. With this, a two-fold passage over any one segment is prohibited. By writing the sequence of symbols appearing during the passage, and eliminating the terms common to each two neighboring segments, one obtains a code ring.

2. Setting up the operating conditions of the feedback circuit. By "operating conditions", we understand those states of the individual counter cells for which the logical feedback path is closed and for which a "one" is written at the circuit's input. These conditions are written in the symbolism of logical algebra in a form convenient for future transformation.

3. Enumerating the unused states, i.e., code combinations which do not enter into the ring.

4. Setting up the function of the feedback path. The function of the feedback path is defined as the logical sum of the operating conditions of this circuit. Moreover, we have the right, with the aim of simplifying the result, to add to this sum any expression from the list of unused states, if by so doing we ease further operations [3].

5. Setting up the functional schematic of the shift register with logical feedback. On the basis of the feedback path function just obtained, and using the rules of logical (Boolean) algebra [4], we set up the functional schematic of the counting device.

6. Verification of the correctness of circuit operation. For this, we write the sequence of operating states and compare them with the sequence of codes of the selected code ring.

We now elucidate what has been said with examples.

Example 1. To develop a feedback circuit for a counter with counting coefficient of 10.

1. We choose any code ring of the seventeen possible four-term binary rings, for example

$$-1001111000- \qquad (1)$$

2. We set up the operating conditions:

$$\begin{array}{ll} 1)\ 1000 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 2)\ 1100 & \bar{a}\bar{b}\bar{c}\bar{d} \\ & \text{or} \\ 3)\ 1110 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 4)\ 0011 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 5)\ 0001 & \bar{a}\bar{b}\bar{c}\bar{d} \end{array} \qquad (2)$$

3. We enumerate the unused states:

$$\begin{array}{ll} 1)\ 1011 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 2)\ 0101 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 3)\ 1010 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 4)\ 1101 \text{ or } & \bar{a}\bar{b}\bar{c}\bar{d} \\ 5)\ 0110 & \bar{a}\bar{b}\bar{c}\bar{d} \\ 6)\ 0000 & \bar{a}\bar{b}\bar{c}\bar{d} \end{array} \qquad (3)$$

4. By writing conditions (2) and adding to their sum the expression $\bar{a}\bar{b}\bar{c}\bar{d}$ from (3), we obtain the following feedback-circuit function:

$$\psi = \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d} = \bar{a}\bar{d} + \bar{a}\bar{b}\bar{d}, \qquad (4)$$

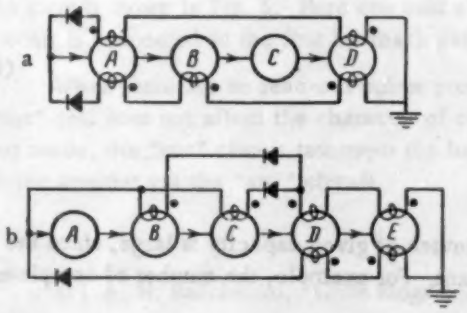

Fig. 4

5. We now set up the feedback circuit from the winding of the shift register elements. The first term of expression (4), $\bar{a}\bar{d}$, can be reproduced by a contrary connection of the windings of cells A and D in series with a diode oriented away from A. It is obvious that there will be a pulse at the output of such a circuit only when there is a pulse in the winding of A and no pulse in the winding of D. The second term, $\bar{a}\bar{b}\bar{d}$, can be reproduced analogously by connecting D counter to A andB, where the diode now must be oriented away from D. Both circuits are connected in parallel. The resulting circuit is shown in Fig. 4, a. The input, output and cycling windings of the cores are not shown on the diagram.

6. We verify the correctness of circuit operation. For this, we give the sequence of working states:

$$\begin{array}{ll} 1)\ 1000 & 6)\ 0011 \\ 2)\ 1100 & 7)\ 1001 \\ 3)\ 1110 & 8)\ 0100 \\ 4)\ 1111 & 9)\ 0010 \\ 5)\ 0111 & 10)\ 0001 \end{array} \qquad (5)$$

We thus convince ourselves that the circuit has ten states, and that their sequence corresponds with the sequence of codes in the selected code ring.

Example 2. To develop a feedback circuit for a 20-cycle counter. Since $2^4 < 20 < 2^5$, the first circuit must be a five-place one.

1. We give ourselves a code ring

$$-11110101001100010000- \qquad (6)$$

We set up the conditions for transmission of a "1" by the feedback path:

a) mandatory

1) 00001
2) 00010
3) 10001
4) 00110
5) 01001
6) 01010
7) 10101
8) 11010
9) 11101

(7)

b) optional

1) 01110
2) 10111
3) 01011
4) 00101
5) 10010
6) 11001
7) 11100
8) 11011
9) 01101
10) 10110
11) 10000
12) 11111

(8)

We set up the feedback circuit function, writting the conditions, for convenience, in the following order (the optional conditions are given in parentheses): 4, (10), 6, 8, (5), 2, (4), 7, 9, (9), 5, (6), 3, 1.

$$\psi = \bar{a}\bar{b}cd\bar{e} + \bar{a}bc\bar{d}\bar{e} + \bar{a}bcd\bar{e} + ab\bar{c}d\bar{e} + ab\bar{c}\bar{d}\bar{e} + a\bar{b}\bar{c}d\bar{e} + \\ + \bar{a}\bar{b}\bar{c}d\bar{e} + \bar{a}\bar{b}c\bar{d}e + \bar{a}\bar{b}cde + a\bar{b}c\bar{d}e + abc\bar{d}e + \bar{a}bc\bar{d}e + \\ + \bar{a}bcde + ab\bar{c}de + a\bar{b}\bar{c}de + a\bar{b}cde + \bar{a}\bar{b}\bar{c}de.$$

After some uncomplicated transformations we get

$$\psi = (\bar{b} + \bar{c})\,d\bar{e} + \bar{d}e. \tag{9}$$

4. We set up the feedback circuit (shown on Fig. 4,b).

5. We verify the correctness of circuit operation.

The number of possible variants of the feedback circuit of a device of given capacity is large, since the number of nonidentical rings of a given length can be quite significant. For example, the number of complete rings (of length 2^n) is determined by the formula

$$p = 2^{2^{n-1}-n} \tag{10}$$

By computing the number of variants of full-length code rings from formula (10), for example, for n = 5, we obtain p = 2048.

Thus, the number of possible feedback circuits for a counter of fixed capacity is very large. Characteristically, among these variants are both very simple circuits and very complicated ones. No one has yet found a universal criterion for choosing the simplest feedback circuit, although it is obvious that the regularity of the code-ring construction immediately affects the complexity of the circuit. There is,as yet,no literature on the synthesis of code rings. Only two sources [2, 5] are known where attempts were made to synthesize code rings, where the method presented in the second source is far from exhaustive.

The presentation, or read-out, of information deserves a brief mention.

Parallel read-out of the codes in all orders of the register is automatically provided at each cycle of circuit operation: each pulse applied to the register's input simultaneously acts on all cells, transferring them to the "zero" state. With this, current pulses arise in the windings of those cores which contained "ones", these pulses supplying information to other blocks of the device. Parallel read-out of information is characteristic of only some computing devices; in the majority of telemetering and remote control devices, the code must be transmitted along a communications line and must therefore be reproduced in the form of a temporal sequence.

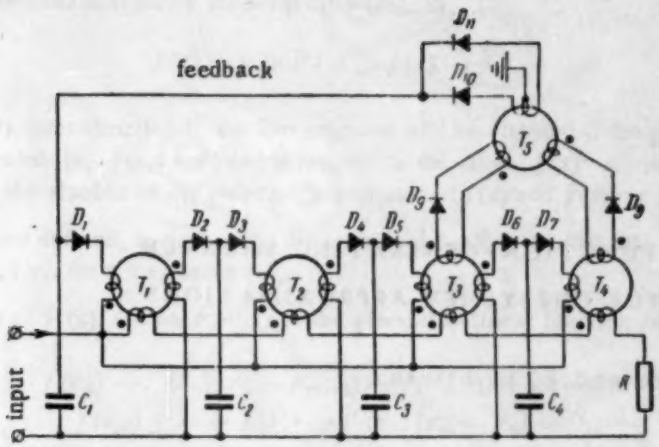

Fig. 5

of staggered pulses. In the given case, a serial code can be obtained by the use of a delay line. For this, the pulses from only one order are applied to the communications channel without delay; the pulses from the other orders are delayed by different time intervals, as a result of which the parallel code is transformed into a serial one.

This method is difficult, however, in the case of large time intervals between code pulses. Therefore, it is more advantageous to carry out reading by introducing a series of pulses into the counter.

For repeated read-out of the same information without disturbing the cores' magnetic states, one can use the circuit shown in Fig. 5. Here one uses a second feedback path containing an "and" circuit, while a "not" circuit is connected in the first feedback path.

When there are no read-out pulses present (these pulses being applied to the "read-out" terminal), the "not" cell does not affect the character of circuit operation, i.e., the feedback signals are passed. In the read-out mode, the "not" circuit interrupts the basic feedback circuit, but the code, which we read out, is rewritten in the counter via the "and" circuit.

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ON CALCULATING FUNCTION-GENERATING SHUNTED POTENTIOMETERS FOR CHEBYSHEV APPROXIMATIONS

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A method is presented for the calculation of a shunted potentiometer intended to reproduce a nonlinear function. The potentiometer is chosen so as to provide the best (Chebyshev) approximation to the function to be reproduced. The design method presented is extended to potentiometers with finite and with infinite load impedances. The method presented in the paper is illustrated by a design example.

Linear potentiometers with homogeneous rheostats and shunting impedances (Fig. 1) are widely used for the approximate reproduction of functions of one independent variable. By controlling the magnitude of the shunt impedances and the places at which they are connected, one can so arrange matters that the error has a Chebyshev character, i.e., so that the following expression is minimized:

$$|\Delta(x)|_{\max} = \max |f(x) - P(x, r_k)|,$$

where $f(x)$ is the function to be reproduced, $P(x, r_k)$ is the function actually reproduced by the potentiometer (the approximating function) and r_k ($k = 1, \ldots, n$) are the parameters of the function-generating potentiometer, i.e., the magnitudes of its shunt impedances.

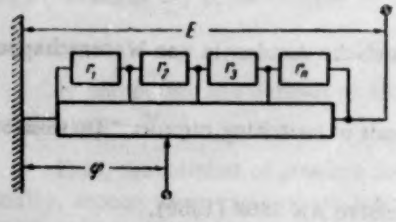

Fig. 1

Two cases of potentiometer design should be distinguished:

a) for an infinite load impedance, the approximating function is piecewise-linear;

b) for a finite load impedance, the problem of a piecewise-nonlinear approximating function arises.

1. The Piecewise-Linear Approximation

The necessity for piecewise-linear approximation arises not only in the design of shunted potentiometers, but also in the case of designing devices with diodes [1]. To solve this problem, it is necessary to determine the parameters of the piecewise-linear function and the boundaries of the pieces. For the minimum number of segments, the approximation must be a chebyshev one and the error is $|\Delta(x)| \leq \epsilon$, where ϵ is a previously given positive quantity. The solutions must encompass two types of functions: with a constant sign of the second derivative (no points of inflection), and with a variable sign of the second derivative in the interval of reproduction.

Let there be given the function $y = f(x)$ which must be approximated by the piecewise-linear function $P(x)$ such that, in the interval $[x_0; x_k]$, the error $|f(x) - P(x)| \leq \epsilon$, and the segments of approximation will be

as large as possible. With the error symetrically distributed on the positive and negative sides, the approximating function must lie within the band defined by the inequality (Fig. 2)

$$f(x) + \varepsilon \geqslant P(x) \geqslant f(x) - \varepsilon.$$

The best (Chebyshev) approximation in the first segment will be attained if the graph of the linear function P(x) is drawn through the point $[x_0, f(x_0) + \epsilon]$ and is tangent to the curve $f(x) - \epsilon$; the boundary, X_1, of the first segment is defined as the abscissa of the point of intersection of P(x) and $f(x) - \epsilon$ (Fig. 2).

Inside the interval thus defined, $x_0 \leq x \leq X_1$, the error is $|\Delta(x)| \leq \epsilon$, and $\max |\Delta(x)| = \epsilon$ occurs on the boundaries of the segment, i.e., for $x = x_0$ and $x = x_1$.

For the constant sign of $f''(x)$, the parameters of the piecewise-linear function are defined by the equations

$$f(x_n) - f'(x_n)(x_n - x_{m-1}) - f(x_{m-1}) \mp 2\varepsilon = 0, \quad (1)$$

$$f(x_m) \pm 2\varepsilon - f(x_n) - f'(x_n)(x_m - x_n) = 0. \quad (2)$$

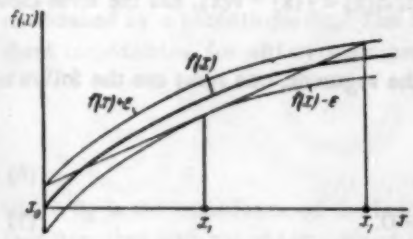

Fig. 2

In these equations, x_n and $f(x_n) \pm \epsilon$ are the coordinates of the points of tangency of the graphs of the linear functions with the boundaries of the error band, $f'(x_n)$ is the slope of the linear function on the unknown segment, x_{m-1} and $f(x_{m-1})$ are the known coordinates of the beginning of the segment of x_m and $f(x_m)$ are the unknown coordinates of the end of the segment.

The subscript m is the ordinal number of the segment (m = I, II, III, . . .) and the subscript n is the ordinal number of the tangency points inside the segments (n = 1, 2, 3, . . .). When the doubled sign (±) is used, the upper and lower signs relate, respectively, to the cases when $f''(x) < 0$ and $f''(x) > 0$

In the general case, Eqs. (1) and (2) are transcendental, and it is necessary to use the method of successive approximations to solve them.

For the solution of Eq. (1), let the rough value $x_n \approx \alpha$ by chosen, and

$$f(\alpha) - f'(\alpha)(\alpha - x_{m-1}) - f(x_{m-1}) \mp 2\varepsilon = \Delta c,$$

where Δc is the discrepancy.

The correction Δx for decreasing the discrepancy Δc is found from the equation

$$\Delta x = \frac{\Delta c}{f''(\alpha)(\alpha - x_{m-1})}. \quad (3)$$

Analogously, one can find the relationship between the discrepancy Δd and the correction Δx for the solution of Eq. (2). Let $\beta \approx x_m$ be an initially chosen value of the independent variable; then the correction is

$$\Delta x = -\frac{\Delta d}{f'(\beta) - f'(x_n)}. \quad (4)$$

In determining the segments of the approximation, it usually transpires that the end of the last one of them does not coincide with the end of the interval of reproduction, i.e., $x_m \neq x_k$. One can achieve coincidence of the boundaries of the last segment and of the interval of reproduction by an insignificant change of the error ϵ over all the segments of the approximation.

Let the value at the end of the last segment be $x = \gamma_1 < x_k$ for $\epsilon = \epsilon_1$. By choosing a new value, $\epsilon = \epsilon_2$, we find that $x = \gamma_2 > x_k$. Then, by linear interpolation, we determine the value ϵ_3 for which $x = \gamma_3 \approx x_k$:

$$\varepsilon_3 = \varepsilon_1 + \frac{x_k - \gamma_1}{\gamma_2 - \gamma_1}(\varepsilon_2 - \varepsilon_1). \quad (5)$$

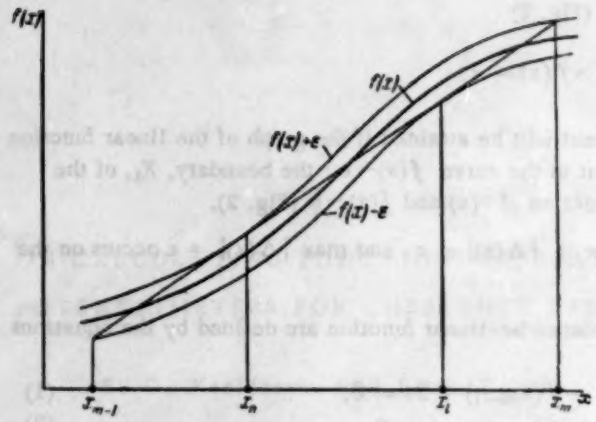

Fig. 3

In approximating a function with a variable sign of the second derivative (Fig. 3), one must choose one of the segments in such manner that the graph of the linear function is tangent to the upper and to the lower boundaries of the error band. The coordinates of the points of tangency of the linear function to the function to be reproduced are thus:

$$[x_n, f(x_n) + \varepsilon];$$
$$[x_l, f(x_l) - \varepsilon].$$

The boundaries of the approximation segment are defined by the abscissas x_{m-1} and x_m. On such a segment, the approximation satisfies the conditions of Chebyshev approximations, and the length of the segment (for a given size of ϵ) is maximal. The error function, $\Delta(x) = f(x) - P(x)$, has the form shown in Fig. 4.

To determine the slope of the linear function and the boundaries of the segment, one must use the following system of equations:

$$f'(x_n) = f'(x_l), \tag{6}$$
$$(x_n - x_l) f'(x_n) - f(x_n) + f(x_l) - 2\varepsilon = 0. \tag{7}$$

From these equations, x_n, $f(x_n)$, x_l and $f(x_l)$ are determined. In the general case, Eq. (7) is transcendental, and the method of successive approximations must be used for its solution. Let the initial value $x_n \approx \alpha$ be chosen. By using (6), we find an intial value of $x_l \approx \beta$. For $x_n = \alpha$ and $x_l = \beta$, the following discrepancy arises

$$\Delta c = (\alpha - \beta) f'(\alpha) - f(\alpha) + f(\beta) - 2\varepsilon.$$

The corrections Δx which allows the discrepancy Δc to be decreased is expressed by the equation

$$\Delta x = -\frac{\Delta c}{f''(\alpha)(\alpha - \beta)}. \tag{8}$$

After x_n and x_l are determined, we can find the coordinates of the left end of the approximation segment by using the equation

$$f(x_n) - f'(x_n)(x_n - x_{m-1}) - f(x_{m-1}) + 2\varepsilon = 0. \tag{9}$$

If, for $x_{m-1} = \gamma$, the discrepancy Δc arises, then the correction

$$\Delta x = -\frac{\Delta c}{f'(x_n) - f'(\gamma)} \tag{10}$$

gives the more accurate value of $x_{m-1} = \gamma + \Delta x$.

The coordinate x_m of the right side of the approximation segment is found from the equation

$$f(x_m) - f(x_l) - f'(x_l)(x_m - x_l) + 2\varepsilon = 0. \tag{11}$$

If, for $x_m = \vartheta$, the discrepancy Δc arises, then the correction

$$\Delta x = -\frac{\Delta c}{f'(\vartheta) - f'(x_l)}, \tag{12}$$

allows one to determine the more accurate value, $x_m = \vartheta + \Delta x$.

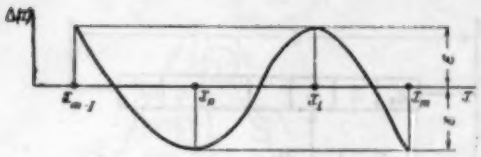

Fig. 4

After one determines the approximation segment in which $f(x)$ has a point of inflection, one determines the approximation segments to the left and to the right by using Eqs. (1) and (2). With this it must be kept in mind that the boundaries of the extreme approximation segments close to the end and the beginning of the given interval $[x_0; x_k]$ do not coincide with the values of x_0 and x_k. By a small change in the magnitude of the error ϵ one can arrange for coincidence of the boundary of one of the extreme approximation segments with the corresponding end of the interval. To obtain coincidence of the boundary of the other extreme segment with the corresponding end point of the given interval, one can decrease slightly the size of the approximating segment which contains the point of inflection of the function being approximated. Such a possibility is based on the circumstance that, in the segment with the point of inflection, the number of greatest deviations equals four (Fig. 4), while for a Chebyshev approximation by linear functions, the minimum number of such deviations equals three.

The material already given allows us to carry out the piecewise-linear approximation of a function to be reproduced by a potentiometer. The following step in the design is the determination of the potentiometer's shunt impedances, for which one must use the equation

$$P(x) = \frac{\varphi}{E} = \frac{R_x}{R}, \tag{13}$$

where R_X is the impedance of the lead-in portion of the potentiometer, R is the total impedance of the potentiometer together with the shunts, φ and E are, respectively, the tapped-off and the applied voltages, P(x) is the piecewise-linear function.

2. Designing a Function-Generating Device with a Finite Number of Parameters

For the approximation of $f(x)$ by a nonlinear function $\varphi(x, r_i)$ $(i = 1, \ldots, n)$, where r_i are the device's parameters, the initial system of parameters id defined by the condition that, at n different points, $\varphi = f$. With this, there arises the error $\Delta x = f - \varphi$. To decrease the error of approximation, one must so vary the initial system of parameters that the value of $\sum_{i=1}^{n} \frac{\partial \varphi}{\partial r_i} dr_i - \Delta(x)$ differs as little as possible from zero. The number n of parameters is finite; from the set of values of $\Delta(x)$ we choose those which correspond to a sufficiently dense series of values of x, contained in $(m-1)$ intervals, where $m > n$. The problem of a Chebyshev approximation reduces to the search for the best solution of the inconsistent system of linear equations

$$\sum_{i=1}^{n} \frac{\partial \varphi}{\partial r_i} dr_i = \Delta(x_\tau) \qquad (\tau = \overline{1, m};\ m > n). \tag{14}$$

In this paper (Cf., the design example), we use as a procedure the so-called gaugeless equalized descent method, developed by E. Ya. Remez [2].

We note that, if the approximating function $\varphi(x, r_i)$ is linear in its parameters, then the initial system of parameters can be chosen arbitrarily. With a nonlinear dependence of φ on the r_i, the procedure mentioned only guarantees such a best approximation, which can be obtained by small variations of the initial system of parameters. Generally speaking, it is necessary to try many initial systems of parameters of the approximating function, each time finding the best approximation by small shifts of the interpolation nodes, in order to guarantee the attainment of the best approximation, which is characterized by the impossibility of decreasing $|\Delta(x)|_{max}$ by any variations of the parameters of the approximation function.

3. Determination of the Approximation Segments and the Choice of the Initial Magnitudes of the Impedances for a Finite Load Impedance

With a finite-load impedance, the output voltage loses its piecewise-linear characters. The load impedance is ordinarily quite large in comparison with the poteniometer impedance. With this, the output voltage actually

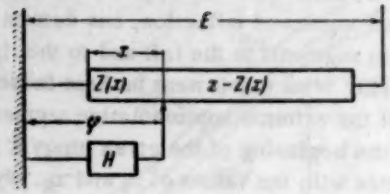

Fig. 5

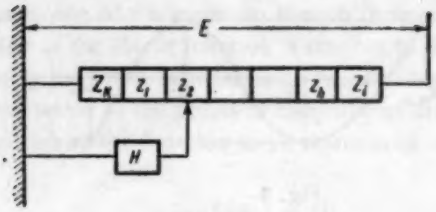

Fig. 6

differs little from a piecewise-linear one, and the method presented in Section 1 can be used for the determination of the approximation segments.

After the approximation segments are determined, the initial values of impedance of the individual rheostats are chosen. In order that the possible load oscillations notshow noticeable influence on potentiometer operation, it is necessary that the total potentiometer resistance z be several times smaller than the given load resistance H.

For a fictitious nonhomogeneous rheostat (Fig. 5), the voltage on the load is

$$\varphi(x) = \frac{EHZ}{HZ + (z - Z)(H + Z)}, \tag{15}$$

and Z is the positive root of the quadratic equation

$$Z^2 + \left(\frac{HE}{\varphi} - z\right)Z - Hz = 0. \tag{16}$$

By setting $\varphi(x) = f(x)$, and then determining Z(x) from (16), we obtain an exact reproduction of the required functional dependence. Since the preparation of rheostats with continuously varying specific resistance (nonlinear potentiometers) is technologically quite difficult, one uses instead rheostats, consisting of several homogeneous portions. With this, the equation $\varphi(x) = f(x)$ will hold only at individual points, called interpolation nodes.

The piecewise-homogeneous potentiometer represented in Fig. 6, with different specific impedances of the individual portions, is usually replaced by a shunted potentiometer (Fig. 7), in which the sliding contact moves along rheostats $R_1, \ldots, R_h$ with identical specific impedances. Impedances r_1, R_1, r_2, R_2, . . . are so chosen that the parallel connection of impedances r_1 and R_1 will be equivalent to resistance z_1, of impedance r_2 and R_2 will be equivalent to resistance z_2, etc.

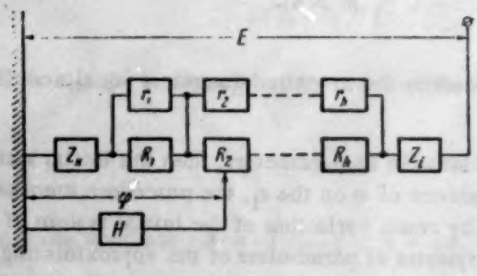

Fig. 7

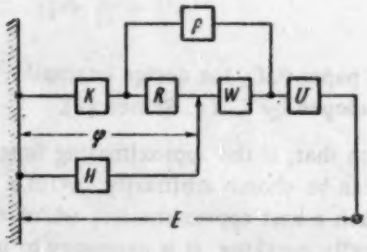

Fig. 8

4. Computation of the Actual Values of Voltage on the Load Impedance and Setting up the Correction Equations

The potentiometer shown in Fig. 6, with the method cited above for choosing the impedances entering into it, will give an exact value of the function to be reproduced when the sliding contact is at any of the segment

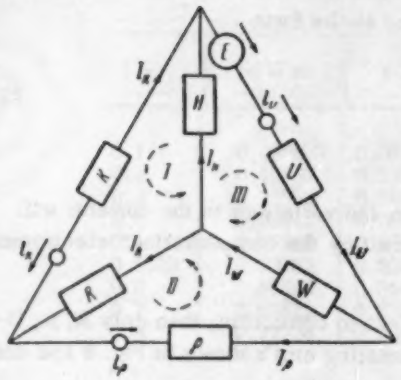

Fig. 9

boundaries. For the other contact positions, the values of the function to be reproduced, $f(x)$, and the approximating function $\varphi(x)$ will not, in general, coincide, and the error $\Delta = f - \varphi$ will arise.

The approximating function φ depends, as will be shown below, on the impedances of the individual potentiometer sections. The parameters of this function which are subject to variation as one attempts to obtain a Chebyshev approximation are assumed to be the magnitudes of the shunt impedances and

$$d\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial r_i} dr_i. \tag{17}$$

To set up system (14) of equations, it is necessary to find an expression for the approximating function φ which gives the values of the actual voltages on load impedance H.

For any position of the sliding contact, the potentiometer can be replaced by the equivalent circuit shown in Fig. 8, where the letters H, K, . . . denote the magnitudes of the corresponding impedances. If the contact is placed as shown on Fig. 7, then $\rho = r^2$, and the sense of the other impedances is clear from a comparison of Figs. 7 and 8.

The computation of the quantity φ can be carried out by the standard electrical engineering method of loop currents. Thus, for example, if one chooses the loopes as shown in Fig. 9, one obtains

$$\varphi = \frac{HE[R\rho + (R + \rho + W)K]}{\Delta}, \tag{18}$$

where

$$\Delta = \begin{vmatrix} R & -H & K \\ R+\rho+W & W & -\rho \\ W & H+W+U & U \end{vmatrix}.$$

Expression (17) can, in general, be obtained by differentiating formula (18), but this way leads to a tremendously complicated expression. A significantly more convenient formula is obtained as a result of giving a physical interpretation to the algebraic relationships given below.

Let there be given a system of linear equations

$$\sum_{j=1}^{n} a_{\tau j} x_j = b_\tau \qquad (\tau = 1, \ldots, n). \tag{19}$$

with a definite solution. If the coefficients of the unknowns are given infinitesimal increments $da_{\tau j}$, then the unknowns will also receive infinitesimal increments dx_j. The new values of the unknowns will be found from the system of equations

$$\sum_{j=1}^{n} (a_{\tau j} + da_{\tau j})(x_j + dx_j) = b_\tau.$$

If we discard second-order quantities form these expressions, and take into consideration that the x_j satisfy Eqs. (19), we obtain the following equation for finding the dx_j

$$\sum_{j=1}^{n} a_{\tau j} dx_j = -\sum_{j=1}^{n} x_j da_{\tau j}. \tag{20}$$

To determine the currents in the loops of an electrical circuit, one sets up the equations which express Kirchhoff's first and second laws. When the loop impedances change, the equation which expresses Kirchhoff's first law does not change. The equations for the second law are transformed to the form

$$\sum Z_j dI_j = -\sum I_j dZ_j. \tag{21}$$

Equations (21) allow the following assertion to be made.

With small variations of the loop impedances in an electrical circuit, the variations in the currents will be approximately equal to the currents, which would be induced in the circuit by the compensating* electromotive forces connected in series with the changed impedances.

If it is assumed that only impedances Z_k, $r_1, \ldots, r_h$ and Z_i are subject to correction, then only K, ρ, U will be varied in the circuit of Fig. 8, and we obtain the scheme of compensating emf's shown in Fig. 9 and corresponding to increases in K, ρ and U.

Now, for the computation of φ, it will only be necessary to compute the current dI_H through impedance H under the influence of the compensating emf's. After this, $d\varphi$ is found from the formula $d\varphi = HdI_H$.

By carrying out the computations, we get

$$d\varphi = \frac{HE}{\Delta^2}\left\{\begin{vmatrix} R+\rho+W & -\rho \\ W & U \end{vmatrix}\begin{vmatrix} R & -H \\ R+\rho+W & W \end{vmatrix} dK + \right.$$

$$\left. + \begin{vmatrix} R & K \\ W & U \end{vmatrix}\begin{vmatrix} R+K & -H \\ R+W & W \end{vmatrix} d\rho + \begin{vmatrix} R & -K \\ R+\rho+W & \rho \end{vmatrix}\begin{vmatrix} -R & H+K \\ R+\rho+W & W+\rho \end{vmatrix} dU \right\}.$$

Formula (22) serves as the basis for setting up correcting Eqs. (14). Computation of formula (22) is greatly eased by the fact that Δ and three of the determinants within the curly brackets were previously computed in the process of determining φ.

APPENDIX

A Design Example

Let it be required to reproduce the function $f(x) = \sin \underline{x}$ in the interval $0 \le x \le 1.4486$ with an error of about 0.01 for a bounded magnitude of load impedance.

On the basis given in Section 1, we determine the four segments:

[0; 0.5323], [0.5323; 0.8814]; [0.8814; 1.1868], [1.1868; 1.4486].

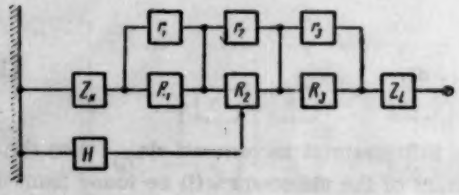

Fig. 10

Since $f(0) = 0$, we shall assume, to simplify the potentiometer, that the sliding contact also moves along impedance Z_k. Then, for the four segments of the approximating potentiometer, we shall have the scheme shown in Fig. 10.

We take H = 50 ohms, the total potentiometer resistance (unloaded) as z = 14 ohms and the voltage applied to the potentiometer as E = 1.04 volts. As the initial interpolation nodes, we take the ends of the segments. By carrying out, under the assumption made, the computations described at the end of section 3, we obtain the following values for the individual impedances:

$$Z_k = 7.311, \quad r_1 = 14.1302, \quad r_2 = 3.3808, \quad r_3 = 0.9160,$$
$$Z_i = 0.507, \quad R_1 = 4.7948, \quad R_2 = 4.1946, \quad R_3 = 3.5957,$$

expressed in ohms.

* The compensating emf in a loop equals the product of the initial current in the loop by the absolute magnitude of the increment of impedance. With an increase in impedance, the compensating emf is directed counter to the initial current, and for a decrease parallel to it.

TABLE 1

x	$f(x) = \sin x$	$\varphi(x)$	$100(f-\varphi)$	x	$f(x) = \sin x$	$\varphi(x)$	$100(f-\varphi)$
0.1	0.09983	0.09956	0.03	0.8814	0.77163	0.77172	—0.01
0.2	0.19867	0.19543	0.32	0.9	0.78333	0.77878	0.46
0.3	0.29552	0.28927	0.62	1.0	0.84147	0.82150	2.00
0.4	0.38942	0.38258	0.68	1.1	0.89121	0.87333	1.79
0.5	0.47943	0.47679	0.26	1.1868	0.92717	0.92719	0
0.5323	0.50752	0.50764	—0.01	1.2	0.93204	0.92792	0.41
0.6	0.56464	0.55456	1.01	1.3	0.96356	0.94209	2.15
0.7	0.64422	0.62686	1.74	1.4	0.98545	0.97192	1.35
0.8	0.71736	0.70350	1.39	1.4486	0.99254	0.99265	—0.01

TABLE 2

τ	x	$a_{\tau 1}$	$a_{\tau 2}$	$a_{\tau 3}$	$a_{\tau 4}$	b_τ
1	0.1	0.458	2.186	4.529	7.130	—0.3
2	0.2	0.905	4.324	8.958	14.103	—3.2
3	0.3	1.357	6.480	13.425	21.135	—6.2
4	0.4	1.825	8.717	18.059	28.431	—6.8
5	0.5	2.324	11.099	22.995	36.201	—2.6
6	0.5323	2.494	11.911	24.676	38.848	0.1
7	0.6	1.776	13.181	27.308	42.991	—10.1
8	0.7	0.754	15.223	31.538	49.651	—17.4
9	0.8	—0.216	17.490	36.235	57.045	—13.9
10	0.8814	—0.963	19.633	40.674	64.033	0.1
11	0.9	—0.928	18.127	41.158	64.796	—4.6
12	1.0	—0.743	10.506	44.114	69.449	—20.0
13	1.1	—0.562	3.584	47.765	75.197	—17.9
14	1.1868	—0.403	—1.925	51.642	81.301	0
15	1.2	—0.390	—1.865	48.361	81.418	—4.1
16	1.3	—0.299	—1.429	25.882	83.236	—21.5
17	1.4	—0.212	—1.011	6.695	86.756	—13.5
18	1.4486	—0.169	—0.809	—1.676	89.167	0.1

In using the initial values just found for the impedances to compute φ, we obtain the results shown in Table 1. Then, by means of formulas (22), we obtain the correcting equations, given in Table 2. These equations were set up under the assumption that impedances r_1, r_2, r_3, and Z_i were subject to variation.

We assume the equations are written in the form

$$a_{\tau 1}\Delta r_1 + a_{\tau 2}\Delta r_2 + a_{\tau 3}\Delta r_3 + a_{\tau 4}\Delta Z_i = b_\tau.$$

where

$$b_\tau = \Delta(x_\tau) = f(x_\tau) - \varphi(x_\tau).$$

In setting up the correcting equations, we used the following expressions for K, ρ and U.

1. The sliding contact on impedance Z_k:

$$K = 0, \quad \rho = \infty, \quad U = \frac{R_1 r_1}{R_1 + r_1} + \frac{R_2 r_2}{R_2 + r_2} + \frac{R_3 r_3}{R_3 + r_3} + Z_i.$$

2. The sliding contact on impedance R_1:

$$K = Z_k, \quad \rho = r_1, \quad U = \frac{R_2 r_2}{R_2 + r_2} + \frac{R_3 r_3}{R_3 + r_3} + Z_i.$$

3. The sliding contact on impedance R_2:

$$K = Z_k + \frac{R_1 r_1}{R_1 + r_1}, \quad \rho = r_2, \quad U = \frac{R_3 r_3}{R_3 + r_3} + Z_i.$$

4. The sliding contact on impedance R_3:

$$K = Z_k + \frac{R_1 r_1}{R_1 + r_1} + \frac{R_2 r_2}{R_2 + r_2}, \quad \rho = r_3, \quad U = Z_i.$$

In correspondence with these, we obtained the following values of dK, dρ and dU on the segments cited.

On the first segment

$$dK = 0, \quad d\rho = 0, \quad dU = \left(\frac{R_1}{R_1 + r_1}\right)^2 dr_1 + \left(\frac{R_2}{R_2 + r_2}\right)^2 dr_2 + \left(\frac{R_3}{R_3 + r_2}\right)^2 dr_3 + dZ_i.$$

On the second segment

$$dK = 0, \quad d\rho = dr_1, \quad dU = \left(\frac{R_2}{R_2 + r_2}\right)^2 dr_2 + \left(\frac{R_3}{R_3 + r_3}\right)^2 dr_3 + dZ_i.$$

On the third segment

$$dK = \left(\frac{R_1}{R_1 + r_1}\right)^2 dr_1, \quad d\rho = dr_2, \quad dU = \left(\frac{R_3}{R_3 + r_3}\right)^2 dr_3 + dZ_i.$$

On the fourth segment

$$dK = \left(\frac{R_1}{R_1 + r_1}\right) dr_1 + \left(\frac{R_2}{R_2 + r_2}\right)^2 dr_2, \quad d\rho = dr^4, \quad dU = dZ_i.$$

The Chebyshev solution of the equations of Table 2 will be

$$\Delta r_1 = 1.2171, \quad \Delta r_2 = -0.0882, \quad \Delta r_3 = -0.0162, \quad \Delta Z_i = -0,1194,$$

in accordance with which we obtain the following definitive values of the corrected impedances:

$$r_1 = 15{,}3473_{\text{ohm}} \quad r_2 = 3.2926_{\text{ohm}} \quad r_3 = 0.8998_{\text{ohm}} \quad Z_i = 0.3876_{\text{ohm}}$$

TABLE 3

x	100δ	100δ*	x	100δ	100δ*
0.1	0.02	0.06	0.8814	1.09	1.13
0.2	—0.22	—0.21	0.9	0.61	0.66
0.3	—0.47	—0.36	1.0	—0.94	—0.91
0.4	—0.48	—0.45	1.1	—0.72	—0.71
0.5	—0.00	0.04	1.1868	1.09	1.10
0.5323	—0.29	0.34	1.2	0.67	0.68
0.6	—0.58	—0.53	1.3	—1.09	—1.08
0.7	—1.09	—0.88	1.4	—0.28	—0.28
0.8	—0.51	—0.48	1.4486	1.09	1.09

With these impedance values, we might, on the basis of the correcting equations, expect the error δ given in Table 3. However, it should be kept in mind that the correction equations give approximate results (since finite, and not infinitesimal, variations of the impedances are made). For this reason, Table 3 also includes the

actual values of the errors δ^*, computed by means of formula (18) on the basis of the definitive values of the impedances. A comparison of the values of δ and δ^* show that they accord satisfactorily with one another.

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TEMPERATURE STABILITY AND ADMISSIBLE VALUES OF POWER AND VOLTAGE ON TRANSISTOR COLLECTORS

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We use the notation: θ_j is the n–p junction temperature, P_t is the power dissipated at the n–p junction, equal to $P_t = V_c I_c$, $R_{t\Sigma}$ is the thermal impedance with heat transfer from the n–p junction. The value of $R_{t\Sigma}$ may be presented (Fig. 1) as

$$R_{t\Sigma} = R_{t0} + R_{tm},$$

where R_{t0} is the thermal impedance for the internal portion (from the n–p junction to the body of the transistor) and R_{tm} is the thermal impedance for transfer of heat from the transistor body to the ambient medium. The increase of temperature θ_j for the n–p junction layer is defined as

$$d\theta_j = R_{t\Sigma} dP. \tag{I}$$

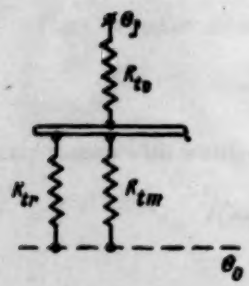

Fig. 1

On the other hand, the power dissipated in the n–p junction layer is defined by the relationship

$$dP_t = V_c dI_c. \tag{**}$$

Since $I_c = I_{c0} + I_{cv}$, where I_{c0} is the portion of colector current which does not depend on the cotnrolling voltage and I_{cv} is the portion of collector current which does depend on the controlling voltage but does not change with temperature, then

$$dP_t = V_c d(I_{c0} + I_{cv}) = V_c dI_{c0}.$$

The dependence of current I_{c0} on temperature is expressed as

$$I_{c0} = I_{c00} e^{\alpha\theta_j} = I_{c00} 10^{\alpha'\theta_j},$$

where the value of α equals: for germanium, $\alpha \approx 0.08$ to 0.85, or $\alpha' \approx 1/30$, and for silicon $\alpha' \approx 1/40$, whence it follows that

$$dI_{c0} = I_{c00} e^{\alpha\theta_j} \alpha d\theta_j.$$

and

$$dP_t = V_c I_{c00} \alpha e^{\alpha\theta_j} d\theta_j.$$

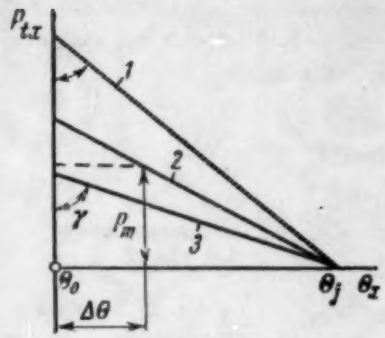

Fig. 2. 1) With radiator for normal air pressure; 2) without radiator for normal air pressure; 3) without radiator for evacuated air; $n_x \tan\gamma / n_y = R_{t\Sigma}$, where n_x is the scale of the axis of abscissas in grams/cm and n_y is the scale of the axis of ordinates, in watts/cm.

Temperature stability for transistors is defined by the condition that the increment of power dP_y dissipated in the n–p junction layer not exceed the increment of power dP transferred through thermal impedance $R_{t\Sigma}$, i.e.,

$$dP_t \leqslant dP.$$

By substituting the values for dP_t and dP, we find that

$$R_{t\Sigma}\alpha V_c I_{c0} e^{\alpha\theta_j} \leqslant 1.$$

As θ_j increases, the left member will be increased. For limiting admissible value, θ_{jli}, we have that the limiting admissible value of collector voltage, $V_c = V_{c\ crit}$ equals

$$V_{crit} \leqslant \frac{e^{-\alpha\theta_{li}}}{\alpha I_{c0}} \frac{1}{R_{t\Sigma}}.$$

With this, R_{tm} must be chosen with the largest value in accord with operating conditions. If there is the possibility of the transistor operating in an evacuated atmosphere, for which R_{tm} increases due to a decrease in the heat transfer coefficient, then it is necessary to choose an R_{tm} which corresponds with the worst heat transfer conditions. The value of θ_{jli} is given on the rating sheet for the given transistor type.

When the load impedance $R_L \neq 0$, the value of V_c is

$$V_c = V_{c0} - I_c R_L,$$

where V_{c0} is the voltage of the collector circuit's power supply. With this, the power dissipated in the collector's n–p junction layer is defined as

$$P_t = (V_{c0} - I_c R_L) I_c,$$

and, consequently,

$$dP_t = (V_{c0} - 2I_c R_L) dI_c = (E_c - 2I_c R_L) dI_{c0}$$

so it will follow, form the condition that $dP_t \leq dP$, that

$$R_{t\Sigma}\alpha (V_{c0} - 2I_c R_L) I_{c0} e^{\alpha\theta_j} \leqslant 1. \qquad (II)$$

It is clear, from a comparison with (I), that the presence of a load impedance in the collector circuit makes it easier to attain the conditions of thermal stability of the transistor.

By knowing the limiting admissible n–p junction layer temperature, θ_{jli}, and the normal temperature of the ambient medium, θ_0, we have the following relationship for the admissible power dissipated in the transistor

$$P_t R_{t\Sigma} = V_c I_c R_{t\Sigma} = \theta_{jli} - \theta_0.$$

In case the temperature of the ambient medium is $\theta_x = \theta_0 + \Delta\theta_x$, then the admissible transistor power is defined by

$$P_{tx} = \frac{1}{R_{t\Sigma}} [\theta_{jli} - (\theta_0 + \Delta\theta_x)].$$

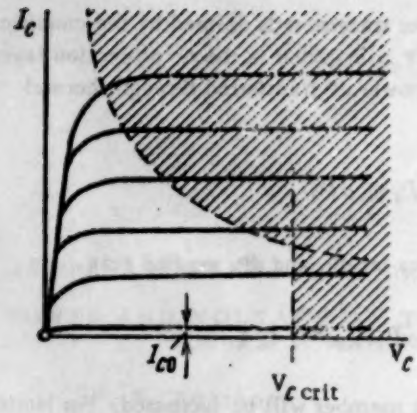

Fig. 3

Figure 2 shows P_{tx} as a function of θ_x for various values of $R_{t\Sigma}$. Mounting a transistor on a radiator is analogous to connecting a secodn thermal impedance R_{tr} in parallel with R_{tm} and, consequently increasing the total thermal resistance

$$R'_{t\Sigma} = R_{t0} + R_{tm}\left(\frac{R_{tr}}{R_{tr} + R_{tm}}\right)$$

and increasing the admissible power P_{tx} for the same values of θ_{jli} and θ_x.

Let the characteristic $I_c = f(V_c, I_b)$ be given for a transistor. Then, by knowing α, $I_{c\,00}$, θ_{jli}, $\theta_x = \theta_0 + \Delta\theta_x$ and $R_{t\Sigma}$, we find the admissible values for:

a) The power dissipated in the transistor

$$P_{tx} = \frac{1}{R_{t\Sigma}}[\theta_j - \theta_x];$$

b) The critical value of collector voltage

$$V_{ccrit} \leqslant \frac{e^{-\alpha\theta_{jli}}}{\alpha I_{c\,00} R_{t\Sigma}}.$$

For this it is necessary to choose the greatest values of $I_{c\,00}$ and $R_{t\Sigma}$ which arise in the given transistor. On the characteristic $I_c = f(V_c, I_b)$ (Fig. 3), we draw the functions $I_c = P_{tx}/V_c$ and $V_{c\,crit}$ = const (dotted lines). The admissible I_c, V_c region for extended transistor operation is bounded by the coordinate axes and the dotted curves just cited.

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A DIGITAL SERVO SYSTEM FOR THE AUTOMATIC PROGRAMMED CONTROL OF A REVERSIBLE ROLLING MILL SCREWDOWN MECHANISM

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A short description is given of a digital servo system for the automatic control of the screwdown device of a sheet steel hot-rolling mill.

The circuits of some of the blocks, as well the formulas of the basic logical cells, are given.

The development of the complex automation of productive processes creates a host of problems related to the programmed control of mechanisms. The programmed control problem arises, in particular, in the automating of the screwdown mechanisms of reversible rolling mills.

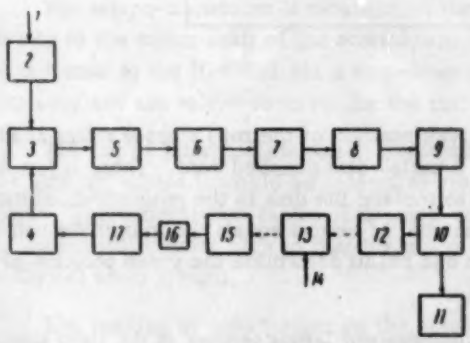


Fig. 1

A system of automatic programmed control of a reversible rolling mill's screwdown mechanism must provide an accurate placement of the mill's upper roller for each pass of th metal in correspondence with the given program of width reduction. There may exist several programs for each rolling mill, depending on the assortment of ingots or sheets to be rolled.

For the automatic displacement of the upper roller, electric servo systems are used, whereby these systems may be either continuous or discrete (digital).

In continuous (analog) servo systems, the actual position of the upper roller is transformed to a continuous electric quantity, for example, to the amplitude of a dc or ac voltage, to phase, etc. The transducers which measure the upper roller's position and transform it to a continuous electrical quantity are ordinarily selsyns, phase inverters, rheostats and inductive potentiometers. Preparation of these transducers with a high degree of accuracy of transformation is very complicated.

For certain types of mills, wehere there are particularly high requirements on the accuracy of upper-roller placement, the use of transducers of the continuous type is only possible when a system of rough and accurate (vernier) computation is used, which significantly complicates the circuitry. The set of programs for screwdown mechanism operation, and also the devices for remembering these programs in the analog systems are very cumbersome and inconvenient. This is particularly essential for mills which operate with a large number of programs.

In digital servo systems, the actual position of the upper roller is transformed to a number, usually expressed in the binary number system, i.e., in a binary code. The program for screwdown mechanism operation is also given in the form of a binary code. The use of digital servo systems allows one to obtain a high accuracy of

screwdown mechanism operation, and to implement the control of screwdown mechanism operation with a minimum of apparatus, as well as a rapid and easy switch-over of programs by means of punched cards.

We give below the description of a servo system, developed by the Central Automation Laboratory of the Construction Ministry of the RSFSR, for the automatic control of the screwdown device of a sheet steel hot-rolling mill. The basic technical requirements to ± 0.1 mm for an operating travel of 400 mm, and the capability of fast and easy switch-over programs for screwdown mechanism operation with a large volume of programs. These requirements conditioned the selection of a digital servo system.

The block schematic of the servo system is shown on Fig. 1, where the following notation was introduced: 1) is the giving of the program by means of a punched card; 2) is the program-furnishing device; 3) is the arithmetic device; 4) transforms the Gray code to an arithmetic code; 5) transforms the difference code to a voltage; 6) is an electronic amplifier; 7) is an electromagnetic amplifier; 8) is the generator; 9) is a motor; 10) is a reducing apparatus; 11) is the screwdown mechanism; 12) is a selsyn-transducer; 13) is a selsyn-differential; 14) is the null correction system; 15) is a selsyn-receiver; 16) is a reducing apparatus; 17) encodes the position of the screwdown mechanism;

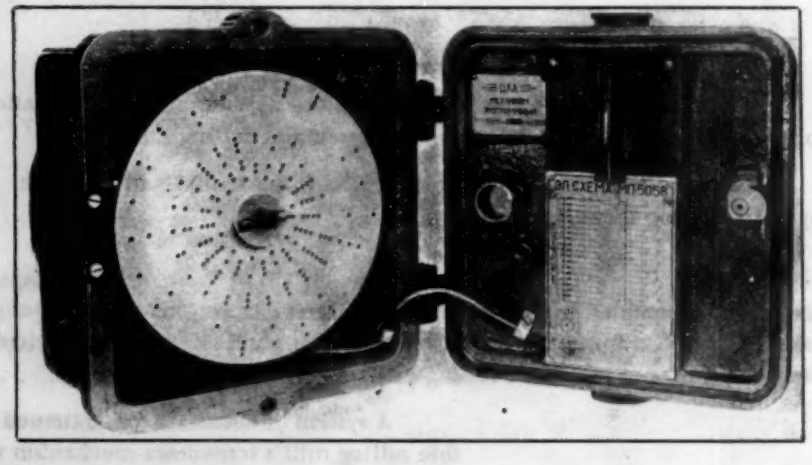


Fig. 2

The program of screwdown mechanism operation, i.e., the given position of the mill's upper roller at each pass, is written in the binary code in the form of round punches on a metal-disk punched card. There is just one program written on each disk. To switch programs, it is necessary to replace the disk in the program-furnishing device (Fig. 2). The disks are divided into sectors by radii. The number of sectors equals the maximum number of passes for rolling mill operation. The holes (punches) placed on one radius determine the given position of the upper roller in the corresponding pass.

For reading the program, germanium photodiodes are used, incandescent lamps serving as the light sources. The photodiodes and light sources are distributed on both sides of the disk, and are maintained immovable. The punched disk, by means of a special stepping mechanism, is rotated one step at each pass, so that a definite radius with punches is positioned between the photodiodes and the light sources.

When a photodiode is illuminated, the magnitude of its back impedance is decreased by several ten-fold. The signals form the photodiodes are amplified and are presented in the form of high and low potentials (with respect to ground) in parallel for all bits of the code. An illuminated photodiode provides a low potential, i.e., 0, to the corresponding bit, while a darkened photodiode provides a high potential, i.e., 1. The number of photodiodes in the program-furnishing device equals the number of bits in the code.

The measurement of the actual position of the rolling mill's upper roller is done by means of a coding transformer. The number of bits in the transformer is determined by the formula

$$n = \log_2 \frac{A}{q}, \qquad (1)$$

where A is the working travel of the upper roller and q is the measurement accuracy.

With the given working travel of the upper roller, it is necessary to use a 12-place (12-bit) transformer to obtain the desired accuracy, corresponding to an accuracy of upper roller measurement of ± 0.025% of the maximum value of the measured quantity.

The use of the binary arithmetic code in the coding transformer could lead to ambiguous transformations due to indeterminacies arising at the boundaries between two discrete neighboring values. Special measures were taken to eliminate ambiguities. One of them is the use of a cyclical progressive code (a Gray code). The distinguishing feature of a Gray code is that only one bit changes when a transition is made from one number to an adjacent one [1] This makes it possible to decrease the errors due to ambiguity in a transformation, involving a low-order bit unit change.

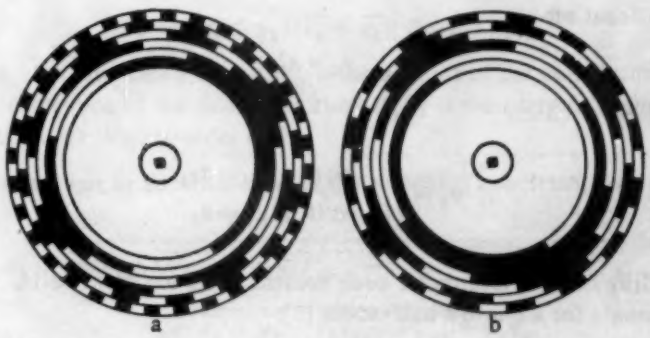


Fig. 3

Figure 3, a shows a disk with a Gray code, containing the six low-order code bits, and Figure 3,b shows a disk containing the higher-order six bits. The disks are prepared from organic glass; the code patterns are placed on the disks by photochemical means. The code disks are connected togher by means of a discrete reducer with a 1 : 32 transfer ratio, and are connected to the screwdown mechanism by means of a selsyn transmitter.

The selsyn-transducer is mounted on the rolling mill housing and, via a reducing apparatus, is connected directly to the motor shaft of the screwdown device (Fig. 1). The selsyn-receiver is located with the code disks, and is joined to the first disk via a step-down reducer. A selsyn-differential is connected between the selsyn-transducer and the selsyn-receiver for the null correction of the servo system.

The use of intermediate selsyn transmission allows the coding transformer to be placed, not on the rolling-mill housing, where it would be subject to vibration, high temperatures and emulsion, but in more favorable conditions. The transmission factor of the step-down reducer between the selsyn-receiver and the coding transformer was chosen sufficiently high so that the errors of selsyn transmission would not lower the accuracy of operation of the digital servo system.

The reading of information on the actual position of the upper roller from the code disks is implemented by germanium photodiodes and the light sources. The signals from the photodiodes are amplified and, for each discrete position of the upper roller, there is presented, in parallel, a combination of high and low electrical potentials in 12 orders (bits). Just as with the furnishing of the program, a low potential corresponding to a 0, and a high potential to a 1.

The actual position of the upper roller, thus coded, cannot be directly compared with the given code, since the Gray code is not arithmetical. Therefore, it is necessary to transform it first to the binary arithmetic code.

The transformer of the Gray code to the arithmetic code, and the arithmetic device itself, consist of the simplest circuits, which execute the logical operations of addition, multiplication and negation. The controlling signals at the input and output of the circuits are combinations of high and low electrical potentials.

The use of potential elements allows us to obtain simple and reliably operating circuits distinguished by high stability and noise stability. The synthesis of the basic logical circuits was done on the basis of the apparatus of the algebra of logic (Boolean algebra). The basic criterion in choosing the circuits was to minimize the number of vacuum tubes, which simultaneously implement the functions of amplification and inversion (logical negation)

of their input signals. In the synthesis a matching condition was adhered to, whereby it was inadmissible to connect an "or" circuit in series with an "and" circuit, since such a connection cannot be carried out with diode elements without the use of intermediate amplification.

We consider now the circuit for transforming the Gray code into the binary arithmetic code. We denote the digits of the successive orders (bits) of the Gray code, in the order of increasing order weight (increasing significance) by $a_1, a_2, a_3, \ldots, a_n$, and the digits of the corresponding order of the binary arithmetic code by $y_1, y_2, y_3, \ldots, y_n$.

The transformation of the Gray code to the arithmetic code is done serially, starting with the most significant bit, in accordance with the following logical rules:

a) for the most significant bit

$$y_n = a_n;$$

b) for the remaining bits

$$y_k = \begin{cases} 0, & \text{ec if } y_{k+1} = a_k, \\ 1, & \text{ec if } y_{k+1} \neq a_k. \end{cases}$$

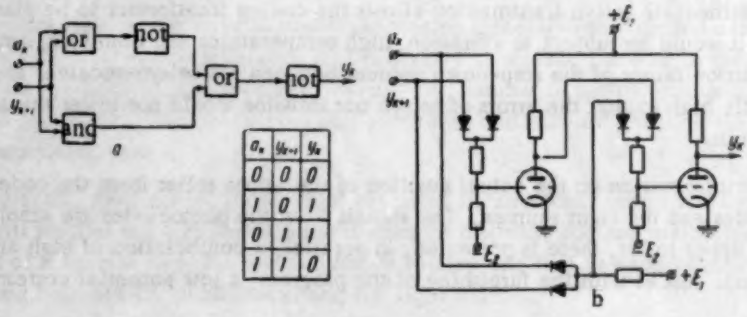
The transformer of Gray code to arithmetic code consists of 12 identical cells. The logical formula of each cell is analogous to the formula for a one-bit half-adder [2]:

$$y_k = \bar{a}_k y_{k+1} + a_k \bar{y}_{k+1}. \tag{2}$$

Use in the transformer of cells constructed in accordance with formula (2) is not possible, since the connection of such a cell with another transformer cell or with the arithmetic device would give rise to the inadmissible connection of an "or" circuit to an "and" circuit.

To obtain the optimal circuit, we transform formula (2) to the form

$$y_k = \overline{\overline{a_k + y_{k+1}} + a_k y_{k+1}}. \tag{3}$$



a_k	y_{k+1}	y_k
0	0	0
1	0	1
0	1	1
1	1	0

Fig. 4

Figure 4 shows the block schematic (a) and the simplified functional schematic (b) of one transformer cell corresponding to formula (3), as well as a table of the logical operation executed by the cell.

A comparison of the given (desired) and actual positions of the upper roller, coded in binary, is implemented in the arithmetic device. The codes to be compared are applied to the arithmetic device's input; at its output, the difference of these codes is obtained. To obtain this difference, subtraction is replaced by addition of the complemented code (of the subtrahend), the complemented code being obtained from the uncomplemented (direct) code by replacing all ones by zeroes and all zeroes by ones.

To avoid additional transformations, the given program is implemented in the complemented code.

Replacement of subtraction by addition of the complemented code makes it possible to use a parallel 12-bit adder in the arithmetic device. Each one-bit cell has three inputs and two outputs. The three inputs are the complemented code from the corresponding bit of the program-furnishing device, the direct code of the actual position from the transformer of Gray code to arithmetic code, and the carry signal from the next lower-significant bit. The two outputs are the sum bit, and the carry bit to the next higher significant bit. The carry signal from the output of the most significant (twelfth) bit cell is applied to the input of the least significant bit cell.

The logical formulas for a one-bit adder has the following form [2]:

$$S_k = [(x_k + y_k)(\bar{x}_k + \bar{y}_k) + P_{k-1}](x_k y_k + \bar{x}_k \bar{y}_k + \overline{P}_{k-1}). \tag{4}$$

$$P_k = [(x_k + y_k)(\overline{x_k} + \overline{y_k})]\, P_{k-1} + x_k y_k. \tag{5}$$

Here, S_k is the sum bit at the cell's output, x_k is the k'th bit of the complemented given (program) input y_k is the k'th bit of the direct code of the actual position, P_{k-1} is the carry from the next lower-order cell and P_k is the carry signal to the next higher-order cell.

Formulas (4) and (5) are not in accordance with the matching condition. We transform them to the form:

$$S_k = \overline{\overline{(\overline{x_k + y_k})\,\overline{P_{k-1}} + x_k y_k \overline{P_{k-1}}} + \overline{x_k y_k + \overline{(x_k + y_k)} + \overline{P_{k-1}}}}, \tag{6}$$

$$P_k = \overline{\overline{x_k + y_k} + x_k y_k + \overline{P_{k-1}}} + x_k y_k. \tag{7}$$

It follows from expression (6) that the inverted (complemented) carry signal from the next lower-order bit must be applied to the input of the current bit. The carry signal to the next higher-order bit must also be inverted

$$\overline{P}_k = \overline{\overline{x_k + y_k} + x_k y_k + \overline{P_{k-1}}} + x_k y_k. \tag{8}$$

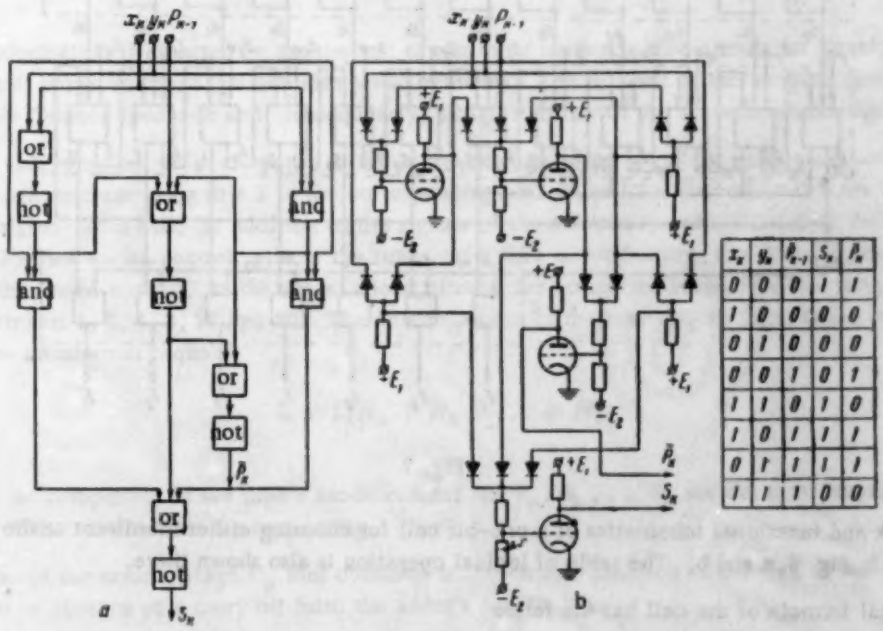
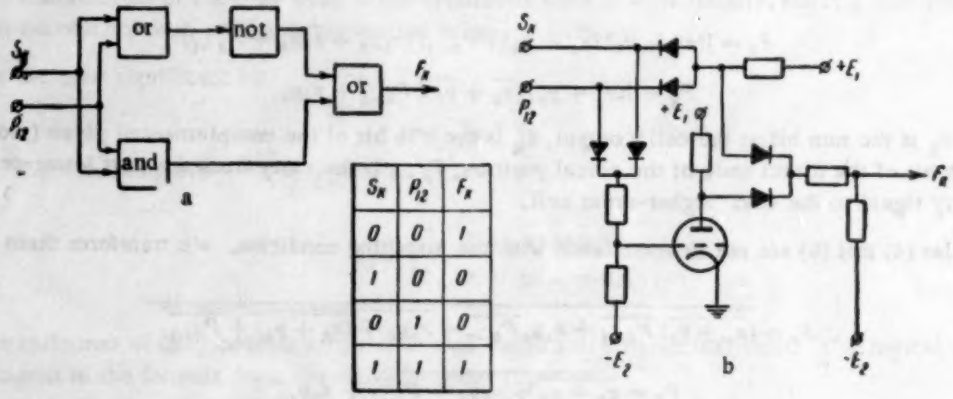


Fig. 5

Figure 5 shows the block schematic (a) and the simplified functional schematic (b) of a one-bit adder, constructed in accordance with formulas (6) and (8). Figure 5 also gives the table of the logical operations implemented by the one-bit adder cell.

With subtraction by the method of complement addition, the difference at the adder's output may be obtained either in the direct or in the complemented code, depending on its sign.

If the difference is positive, then it is given in the direct code, and there is a carry signal from the output of the most significant (twelfth) bit of the arithmetic device. If the difference is negative, it is given in the complemented code, and there is no carry signal. Thus, the presence or absence of a carry from the most significant bit indicates both the sign of the difference and the code in which it is expressed. If the difference is obtained in the complemented code, it must be transformed to the direct code.



S_k	P_{12}	F_k
0	0	1
1	0	0
0	1	0
1	1	1

Fig. 6

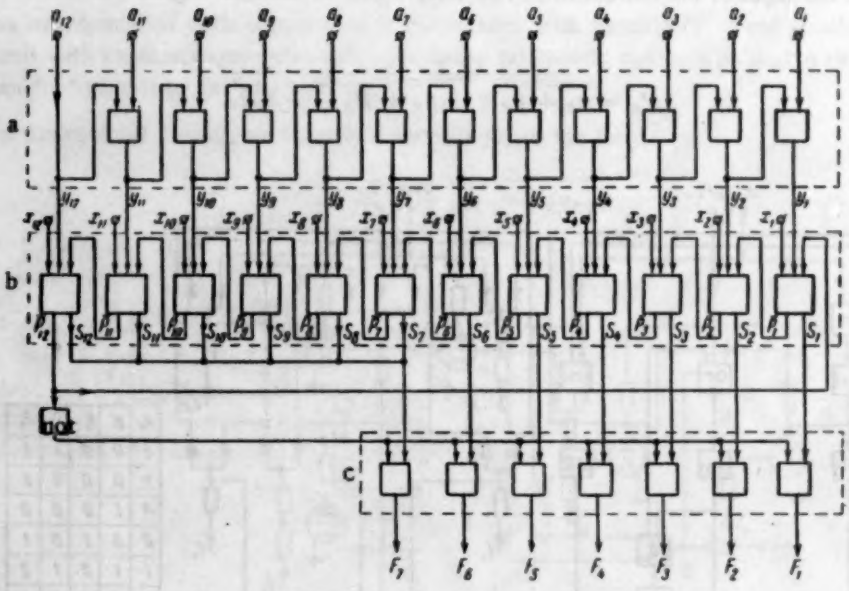


Fig. 7

The block and functional schematics of a one-bit cell for choosing either the direct or the complemented code are shown in Fig. 6, a and b. The table of logical operation is also shown there.

The logical formula of the cell has the form:

$$F_k = S_k P_{12} + \overline{S_k + P_{12}}. \qquad (9)$$

Here, F_k is the output bit of the code choice cell, S_k is the sum bit at the output of the adder's k'th bit and P_{12} is the carry bit from the highest-order (twelfth) bit of the adder.

The block schematic of the logical circuit of the digital servo system is given in Fig. 7, where a shows the transformer of Gray code to arithmetic code, b shows the adder and c is the circuit for choosing between the direct and the complemented code.

For the control of the electric drive system, the difference code obtained is transformed to a sinusoidal voltage whose amplitude depends on the discrepany (i.e., the error).

Stability, and the required transient response quality in the system, are attained by introducing flexible feedback paths in continuous (analog) form. For this, it suffices that the controlling voltage depend linearly only on those errors which are close to zero. Because of this, only the six lowest-order bits are transformed to a proportional voltage, this corresponding to an error somewhat in excess of the braking travel of the screwdown device motor when oen brakes from the maximum velocity to zero in the case when the maximum braking torque is applied.

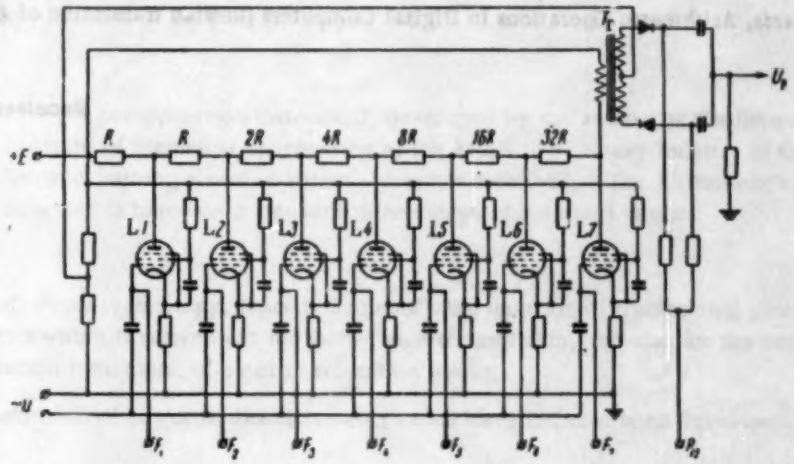


Fig. 8

The transformation is effected by the method of summing currents. The circuit for transforming difference code to voltage is shown in Fig. 8. It contains seven pentodes. The resistors in the vacuum tubes' cathode circuits provide the flexible feedback and, consequently, a high stability of the ac components through the pentodes.

Tubes L1-L6 being cut off corresponds to a 0 in the corresponding bit of the difference code, the conducting of any of them corresponding to a 1 in the corresponding bit. Tube L7 will conduct if there is a 1 in at least one of the six higher-order bits. In addition to the signals of the difference code, sinusoidal voltages of identical amplitude are applied to the control grids of the tubes. If a tube is conducting, this voltage determines an ac component in the anode current. In the tubes' anode circuits are connected resistors whose magnitudes are multiples of the numbers 1, 2, 4, 8, 16 and 32. The ac component of the resulting voltage, taken off the resistors by means of the transformer, equals

$$u = i\,(R_a + R_b + \ldots + R_k), \tag{10}$$

where i is the ac component of the tube's anode current and R_a, R_b . . ., R_k are the total impedances in the anode circuits of the tubes which are conducting.

The phase of the error voltage U_p thus obtained is chosen as a function of the sign of the error, determined by the presence or absence of a carry bit from the adder's twelfth bit.

A constant potential, equal to half the voltage corresponding to a 1 at the adder output, is applied to the middle of the transformer's secondary winding. The carry signal from the twelfth bit equals 0 or 1, corresponding to a high or a low potential, and permits a voltage of the necessary phase to be obtained in the load on the transformer's output.

The error voltage thus obtained is applied to a phase-sensitive electronic amplifier which forms the voltage which controls the electric drive system.

The operation of all the system blocks was verified on mock-ups. They are simple to adjust, very non-critical to variations in circuit parameters, stable under external noise and operate stably when the supply voltage varies by ± 15%.

The digital servo system herein described can be used, not only for the automatic control of rolling mill screwdown mechanisms, but also for other program-controlled mechanisms.

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A COMPENSATION INSTRUMENT FOR TRANSFORMING POWER TO DIRECT CURRENT

M. Duma and M. Syrbu

(Bucharest)

A compensation instrument, developed by the authors at the Research Institute of Electrical Engineering of the Ministry of Heavy Industry of the RPR, for transforming power to direct current, is described. The instrument's construction is based on a standard three-phase, three-stage meter.

In power telemetry technology, there is frequently the question of tranforming power to direct current, both as a parameter which is convenient for short-range transmission, and also for the summation, and transmission of the summation indicators, of electric substation power.

In the transformers of power to direct current, which have hitherto been developed, the following properties are inherent.

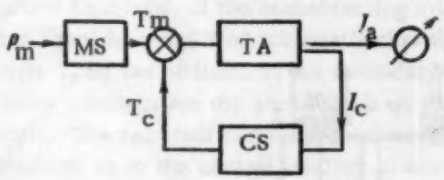

Fig. 1

a) The existing static transformers [1-4] can provide accuracies of from 1 to 1.5%, but possess a number of disadvantages. For example, the thermoelectric transformers [2; 3], provide low dc power levels at their outputs, so that special amplifiers are required if they are to be used in short-range telemetry devices. The magnetic transformers [1, 4] require that complex operations of sorting, calculating and regulation be implemented, which is not readily applicable to mass production.

b) The existing transformers which contain moving parts, operating in closed control loops, for example, in accordance with the principle of equalizing the active torque induced by a three-phase wattmeter with the counteracting torque produced by a magnetoelectric milliammeter, can provide the necessary output power for long-range transmission of indications, but require specially developed accurate mechanical construction, which is undesirable. The accuracy of these devices depends on the accuracy with which the mechanical construction is implemented.

In 1957, the authors, working in the Research Institute of Electrical Engineering of the Ministry of Heavy Industry of the RPR, developed a compensation transformer of power to direct current, with class 1 accuracy and allowing telemetry of indications to be made over distances of tens of kilometers over wire communications channels. The transformer is easy to reproduce and is of simple construction, since it is constructed of standard elements of large-scale mass production.

The principle of operation of the developed transformer of power to direct current (Fig. 1) consists in the equalization (balancing) of the active measured torque T_m, generated in the two measuring systems MS of an induction meter, by a counteracting torque T_c, generated by means of transformer amplifier TA in the counteracting system CS of the same meter. The counteracting torque T_c follows the changes in T_m and tends to equalize (balance) them. As is generally known, the torque developed by the moving system of an induction instrument

with moving poles is proportional to the power measured:

$$T_m = K_m P_m \text{ where } P_m = U_m I_m \cos\varphi_m,$$
$$T_c = K_c P_c \text{ , where } P_c = U_c I_c \cos\varphi_c.$$

Here the subscripts "m" and "c" denote the quantity relative, respectively, to the measured and counteracting systems.

For a balanced system, the counteracting torque equals the active measured torque, i.e., $T_m = T_c$. If the voltage U_c of the counteracting system is maintained constant, and the angle φ_c equals zero, then the measured power is proportional to the actual value of the counteracting current:

$$P_m = \frac{K_a U_a}{K_m} I_a$$

After rectification, the average current value I_a at the output of the transformer amplifier will be proportional to the effective value of the ac current I_c of the current winding of the counteracting system. In the case of two-half-period rectification, if we consider K_i as the transformation coefficient of current transformer T, we obtain

$$I_a = \frac{4}{\pi K_i} I_a$$

Then,

$$P_m = \frac{\pi K_a K_i U_a}{4 K_m} I_a.$$

As is seen from the last expression, the magnitude of the direct current at the rectifier's output is proportional to the value of the measured power.

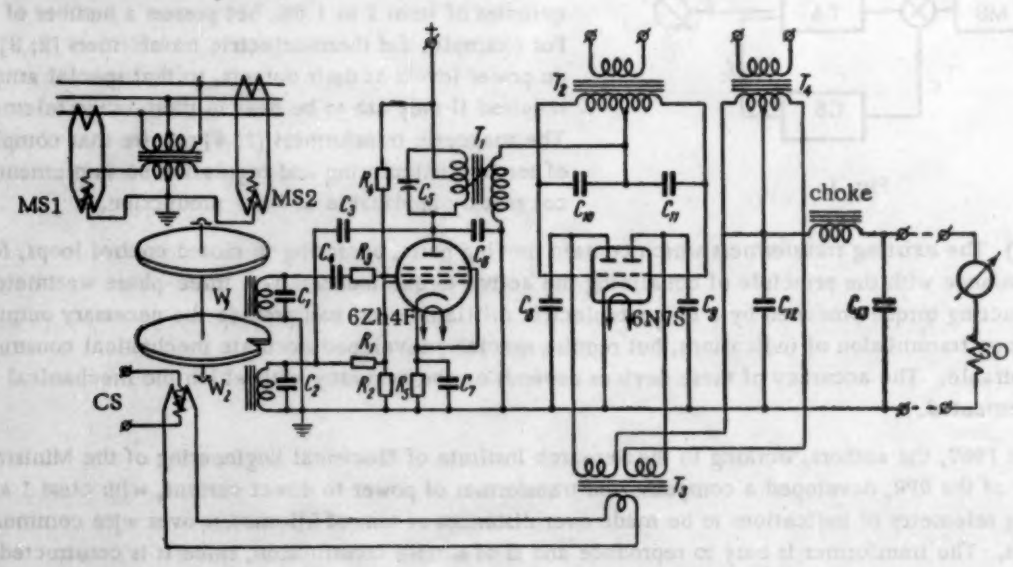

Fig. 2

To derive these formulas, we made the following assumptions.

1. The expression $U_c \cos\varphi_c$ is constant even with oscillations of the line voltage over the entire range of transformer operation: this can be accomplished by stabilization of the value of $U_c \cos\varphi_c$ or by stabilizing only the effective value of voltage U_c of the counteracting system for an angle φ_c between U_c and I_c approximately

equal to zero, such that cos φ_c is alwyas close to unity. Then, small variations in the angular shift between the two vectors during system operation do not have significant effect.

2. Over the whole range of transformer operation, the coefficient of proportionality, K_2, between the average value of output current I_a and the effective value of counteracting current I_c remains invariant. This means that, over the entire range of transformer operation, the transformation coefficient K_1 of current transformer T_3 (Fig. 2) remains invariant.

3. There are no distortions of the form of the counteracting current curve which would disturb the proprotionality between the average value of the current and the effective value of its first harmonic component. The counteracting system's voltage winding is supplied by a virtually undistorted line voltage, so that only the first harmonic induces a counteracting torque.

4. Over the entire transformation range, the ratio K_c/K_m remains constant. In actuality, the error curve of an induction meter for variations in load is analogous for each system. Consequently, the changes of coefficients K_c and K_m with the load follow the same law, so that the ratio K_c/K_m can be considered constant.

Figure 2 shows the simplified schematic of the compensation transformer of power to direct current. As the comparison element, we used a standard three-phase, three-stage electric energy meter in which the necessary changes were made; the first of these was the removal of the metering mechanism.

Measurement of three-phase power is implemented by the method of two wattmeters in measuring systems MS1 and MS2. On the meter's moving system there acts a rotational torque proportional to the measured power. Windings W_1 and W_2, together with capacitors C_1 and C_2, comprise the tuned circuits of a high-frequency generator. A notch was made in the lower disk, providing an increase in the coupling factor between the two windings when the disks are rotated to the side corresponding to the effective active torque, induced by systems MS1 and MS2, by which the amplitude of the auto-generator's high-frequency oscillations is correspondingly varied.

When the measured power equals zero, there are no high-frequency oscillations, and output tube 6N7S is cut off by the voltage from transformer T_2 which is in phase opposition to the anode voltage. When an active torque from the measured power appears, thus generating high-frequency oscillations, tube 6N7S conducts to the extent required to induce, in the counteracting system CS, the ac current I_c necessary to equalize the torques. It is clear from the simplified schematic that the dc current of the counteracting system is produced in current transformer T_3 by the addition in the secondary winding $\underline{x}$ of the two half-wave currents in the primary windings. The primary windings are the anode loads on the two halves of tube 6N7S, operating in a rectifier-controlled two-cycle circuit. The rectified current thus obtained is filtered and directed to the communications channel for power telemetry, or to the control winding of summing device SD if summing is carried out at the sending points.

It is clear from the foregoing exposition that the transformer developed is a balanced-current instrument. In such devices, one eliminates almost completely the errors stemming from variations in circuit parameters, line parameters, voltage supplies, aging of the electronic tubes of the transformer's electronic amplifier, etc. Since, the instrument is astatic, the error rapidly tends to zero if any such external excitation appears.

The physical construction of the transformer is comprised of two blocks (Fig. 3): the compensator block, with dimensions of a three-phase meter, and the electronic block.

Testing of the compensation power transformer showed that basic instrument error did not exceed $\pm 1\%$ this error being the deviation of the characteristic from linearity with changes of the input quantity from zero to 120% of its nominal value, when the power coefficient was $\cos \varphi_c = 1$, the frequency was $f = 50$ cycles and the temperature was $t = 20°C$. The graph on Fig. 4 is the average of 40 measurings, carried out during a period of 200 days. The axis of abcissas give the value of the measured power (in percentages of its nominal value) and the axis of ordinates given the basic transformer error (in %). The maximum value of the error was 0.8%.

The additional errors when the temperature varies by $\pm 20°C$, the supply voltage varies by $\pm 15\%$, the frequency varies by $\pm 2\%$, the communications line impedance varies from zero to 1500 ohms, and the distortion coefficient of the supply voltage varies by 5% do not exceed 1% each.

With a two-disk meter construction, the additional error from the variations of $\cos\varphi$ amounts to 1.5%; with a three-disk construction it is less than 1%.

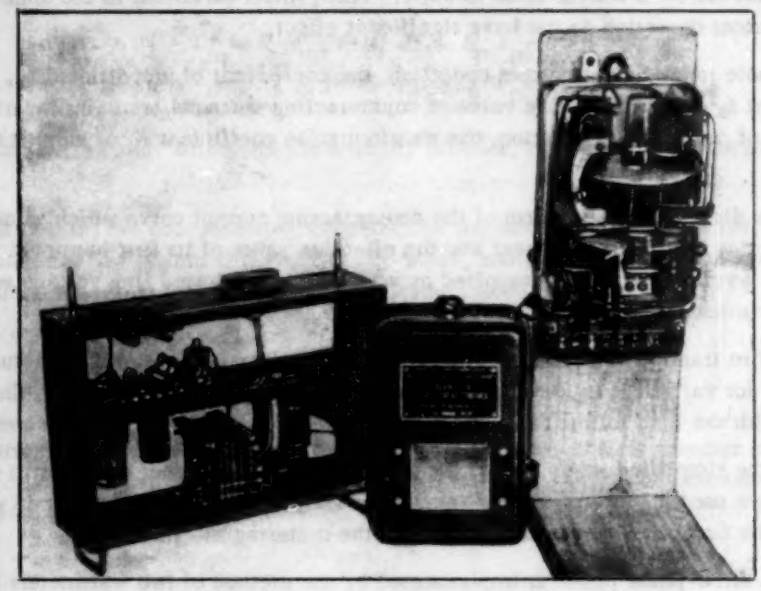

Fig. 3

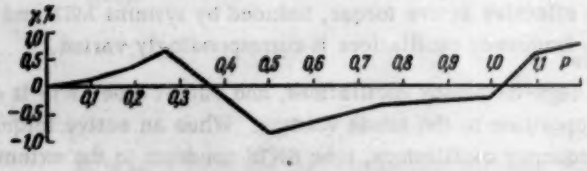

Fig. 4.

The time to establish the output current when the input power changes instantaneously from zero to its nominal value, does not exceed 2 seconds; the oscillatory process (the transient response) is rapidly damped. Such a rapidly-damped transient response is of great value in the case of short-range telewattmeters, since the frictional error is such an instrument is reduced.

The transformer requirements correspond to the standards for meters.

For different variants of the physical construction, the dc current at the transformer's output, for U_C = = 100 volts, I_C = 5 amps and $\cos \varphi_C = 1$, was taken to be from 5 to 20 milliamps.

The transformer continued to operate satisfactorily under difficult operating conditions: for a change in the communications line impedance from zero to 4000 ohms, for a 40% increment of the input power from its nominal value, with a distortion coefficient up to 10%, the additional errors were of the order of 2%.

In the case of single-phase power measurement, each transformer can be used for summing two indications.

The necessity of stabilizing the counteracting voltage might be considered a disadvantage of the transformer. Since each stabilizer services four transformers, the cost of the devices is increased insignificantly. The transformer can also work off a nonstabilized source, with the condition that the controlling or summing devices are supplied by the same source as the transformer. The errors attributable to these devices when the voltage changes must be equal to the transformer errors, but opposite in sign.

The basic disadvantages of the described electronic-induction compensation transformer of power to direct current are the following.

1. Simple construction, based on standard three-phase meters. The complex artifacts of exact mechanics are not necessary, which makes production of such transformers much more economical than the production of other existing instruments with moving parts, operating with closed, or with open, control circuits.

2. Favorable conditions for mass production, since the transformer contains only easily adjustable and reproducible elements, which do not require complicated individual tuning, sorting or measuring, or individual design.

3. The basic error (i.e., deviation from linearity) is less than 1% for the transformer of power to direct current.

Such accuracy can be obtained even with the use of class 2.5 meters, because [7]:

a) removal of the metering mechanism significantly decreases the frictional errors;

b) a mutual compensation of the nonlinearity curves of the measuring and counteracting systems is possible, since they are analogous;

c) when a meter is used in a transformer of power to direct current, the error is referred, not to the given measured quantity, as is the case for electric energy meters, but to the nominal value of the power.

4. Large output power, allowing the transformer to be used as a short-range telewattmeter without an intermediate amplifier.

5. There is no connection between the moving and the immovable systems of the instrument, as there are in all other existing transformers with mobable parts, due to flexible filaments, springs, brushes, etc.

6. Reliability in use, provided by the absence of easily maladjusted elements and by the entire physical configuration of the transformer.

7. The possibility of signalling or controlling protective elements if the supply voltage fails, the counteracting torque disappears, the oscillations are disrupted or there is an external defect, for example, if the communication line is cut.

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CRITICISM

REVIEW OF THE BOOK "INTRODUCTION TO AUTOMATIC CONTROL THEORY" BY A. YA. LERNER

(Mashgiz, 1958)

The book contains a short course in the fundamentals of automatic control theory. It is intended basically for the broad circle of engineers who work in the domains of automation of production, the development of automata circuitry, the installation and adjustment of automatic control systems.

From the large number of questions contained in automatic control theory, the most basic, a knowledge of which is necessary for the types of readers cited above, have been included in the book. The author divides the book into three parts, unequal in size.

The first part of the book acquaints the reader with the nomenclature of automatic control systems, their structures, with the basic technical ideas connected with the use of negative feedback, with the terminology and the classification of automatic controllers. The role of automatic devices in the control of productive processes is explained, and the basic information, necessary for the static and dynamic analysis of an automatic control system, is provided. Unfortunately, this portion does not contain descriptions of the standard apparatus used in automating production. The author seemingly assumed that descriptions of such apparatus must be contained in courses on the technical means of automation, but such courses do not exist, and for the reader referred to above, it would certainly be useful to include in this section basic information about the standard apparatus, their characteristics and capabilities for use in various circuits.

The second part is the basic portion of the book, and is devoted to the statics of automatic control systems, to the dynamic consideration of control systems, chiefly in linear approximations, and to certain data on the properties of nonlinear automatic control systems. This book usefully distinguishes itself from other textbooks containing approximately the same material by its detailed consideration of the questions related to the static relationships of the system "controlled object – controller".

The dynamic investigation begins with the classification of linear links and with general information about the algebra of linear circuits and about the different types of frequency characteristics (including logarithmic characteristics). The chapter devoted to stability contains an exposition of the basic stability criteria (Hurwitz, Mikhailov, and phase-amplitude criteria) and a brief discussion on the construction of the regions of stability by the method of D-decomposition. Since the Mikhailov and frequency criteria of stability are presented earlier than the D-decomposition criterion, the proofs of these criteria is carried out independently of the D-decomposition method, being based, essentially, on principles which are not explicitly introduced.

The chapter devoted to methods of analyzing transient responses is clearly inadequate, both in volume and in the amount of material contained in it. In particular, a considerably more detailed consideration was deserved by the integral estimates and by the estimates related to the distribution of zeroes and poles (some space was allotted to these questions in the book's third chapter).

With respect to the estimates of the form of a transient response from the real characteristic, the author's presentation corresponds approximately to that adopted in contemporary courses, although it has recently, particularly in the works of Blokh, been shown that one can obtain more complete informtion about the transient response, as well as more simply, by the use of the imaginary characteristic.

The third part, the most interesting in the book, is entitled by the author, "Determination of the Structure and Calculation of the Parameters of an Automatic Control System". Essentially, it considers the problems of

synthesizing linear-control systems, plus certain problems related to the creation of optimal nonlinear systems. The first chapter of this part (Chapter 10 of the book) is devoted to the questions of greatest importance to designers, and almost completely ignored in textbooks on automatic control theory – the selection of automatic controllers, and their adjustment to accord with the data on the controlled object and the technical data on the control process. The following chapter considers the more general questions, which are also treated in other monographs and course, of synthesizing stabilizing and correcting loops for linear systems. However, in contradistinction to other courses, where only frequency methods are used for this purpose, the author also adduces the root hodograph method. Many works are devoted today to this method, principally in the United States. In the Soviet Union, unfortunately, only the work being reviewed treats of this method, and the author's attempt to familiarize the afore-mentioned group of reads with the basic ideas of this method should certainly be applauded. The last chapter is also a novel feature of the book, since the questions treated in it have, until now, been considered only in scientific works. It is devoted to the theory of systems which have optimal speeds of response. In relation to this, the author introduces the basic concept of a process which is optimal as regards its speed of response; the reader is familiarized with the most fundamental ideas of implementing optimal processes by means of nonlinear connections. Two basic remarks may be made anent this section. One deals with the chapter devoted to the selection of controllers. The fact that only 12 pages of the book are given to this chapter speaks for itself, and shows that the space which this chapter occupies in the total volume of the book does not correspond to the tremendous importance of the questions discussed in the chapter. In designing a scheme for automating production, the engineer-designer has far more frequently to solve the problem of how to select the controller, and the engineer-adjuster, how to tune it, then either of them has to carry out a dynamic, or even static, calculation of the system. Certainly, it is difficult to make recommendations on choice of controller, since not enough factual material, unfortunately, has been accumulated for this by automatic control theory, but the book's author is also well known for his work in practical domains, and he could be required to have devoted greater attention to this question.

The second remark has to do with optimal system theory. Essentially, the author limited himself to presenting his own investigations in optimal system theory. They are interesting but, since, they were published, a number of first-class works have appeared, both in Soviet Union and in the United States, in the direction of the development of optimal system theory. Certain of them, for example, Bellman's method of dynamic programming, can be presented in sufficiently approachable form to be useful for the circle of readers for whom the book is intended.

As a whole, the book is a necessary and useful addition to the courses on automatic control theory which had appeared previously.

The book contains individual inaccuracies and typographical errors. Thus, for example, an objection arises to the classification of automatic control systems given in § 1 of Chapter 3. It is impossible to concur with the estimate of the effectiveness of introducing higher derivatives in integral estimates, made by the author of Chapter 11 (É. G. Uderman). Misunderstandings can arise due to the typographical errors and to the inaccuracies in the diagrams of which the book is not free due, apparently, to a lack of care in its editing. It may be hoped that these inadequacies will be eliminated in later editions of the book, which, certainly, must shortly follow, since the need for a book on automatic control theory which is accessible to practical workers has yet to be satisfied.

V. Yu. Kazanov

REVIEW OF THE COLLECTION "LOGICAL INVESTIGATION"

(Academy of Science of the SSSR, Philosophy Institute,
Izdatel'stvo AN SSSR, Moscow, 1959)

In digital technology today, particularly in automation and in remote control, broad use is made of the methods of mathematical logic, which turns out to be a tremendously convenient apparatus for solving the problems of analysis and synthesis of discrete (switching) devices. In the collection reviewed here, devoted to the problems of contemporary logic and its scientific applications, a significant amount of space is allotted to the technological applications of mathematical logic. The collection contains a number of papers, a familiarity with which would be of interest to engineers and to scientific workers occupied with the problems of digital automation and remote control. In particular, these papers contain valuable results in the analysis and synthesis of switching circuits and discrete automata, obtained by means of mathematical logic.

The collection consists of two parts. The first part is devoted to general questions of symbolic logic, its history and its particular applications to the new physics, to linguistics and to other domains. We shall here consider in detail only the second part. It contains nine papers relating to "technical logic". In addition, three papers are contained here which are devoted to the logical problems of the foundation of mathematics. In the paper of A. D. Getmanova, "On the relationship of mathematics and logic in systems of the type of the 'Principia Mathematica'," there is presented the justification, developed by the author, of the negative solution to the problem of the reduction of mathematics to logic, in connection with which the author deals with certain characteristic features of logic and, in particular, discusses the question of theoretical modes of logical systems.

In his long paper, "On potential realizability," which warrants special discussion, A. S. Esenin-Vol'pin presents the "outspoken", to use the author's terminology, point of view on realizability in mathematics. This point of view requires that one take into account, not only the finitude of a number, but also its pratical attainability, which might turn out to be very important for the arithmetization and algorithmization of various information-processing schemes.

Finally, in the paper of V. S. Chernyavskii, the concept of "shuttle" algorithms is considered, and a proof is given of the equivalence of shuttle and normal (in the sense of A. A. Markov) algorithms and Turing machines.

The "technicologic" series of papers is opened by the paper of G. N. Povarov, "Logic and automation" which acquaints the reader with the place and value of logic in the automation of production. There are discussed in it the fundamental problems of technological logic, the history of the application of logic to technology, and the basic results attained by Soviet and foreign scientists in this domain. The author mentioned that the introduction of the results of technological logic into the operations of planning organizations and into the teaching of thecnical disciplines in the higher schools will promote technological progress in our country and increase its scientific-technological power. The paper, written in popular style and with the rich bibliography removed, could serve as a good textbook for those who might wish to enter upon the circle of questions connected with the technological applications of logic.

The two other papers of G. N. Povarov are of a different character. The first, "Mathematico-logical investigation of the synthesis of switching circuits with one input and k outputs," is a paper given by the author at an international symposium on switching theory, held at Harvard University in 1957. As is well known, this paper was highly praised in the foreign publications. In it are investigated, by means of mathematical logic and combinatorics, the taxonomy and complexity of switching (1, k)-poles, and a number of practical methods for

synthesizing them are given, based on Boolean algebra or its graphical interpretations, in particular, the method of cascades. It should be mentioned, that this method underlies the machine for synthesizing switching circuits which was constructed at the Laboratory for Information Transmission Systems of the AN SSSR and was successfully demonstrated at the Brussels Fair of 1958 and the New York Exposition in 1959. The paper devotes particular attention to symmetric and other ordered schemes.

The second paper, "On the logical synthesis of electronic computers and controlling sytems," contains several methods for synthesizing electronic switching (n,k)-poles, which are subdivided into converging and diverging. The first are analogous to series-parallel switching circuits, the second to bridge circuits. The paper contains descriptions of the tree method, the cascade method, a strengthened cascade method and a method analogous to the method of universal multipoles for switching circuts. The logical basis of the synthesis is the presentation of some function of logical (Boolean) algebra as the superposition of elementary functions, which are realized by cathode followers, triodes, and pentodes.

The paper of V. I. Shestakov, "Simulation of the operations of the propositional calculus by means of switching circuits," consists of two parts. The fundamental content of the first part is the establishment of the isomorphism between the propositional calculus and the algebra of pi circuits. In addition, the principles of contact and switching circuit realizations of the formulas of the propositional calculus are described.

In the paper's second part, the author turns from the four-terminal realizations of two-position switches to the realizations of three-position switches. Then, the class of three-position (three-valued) switching circuits is considered, one of the constructive properties of this class being used for the modeling of the operations of D. A. Bochvar's three-valued propositional calculus. The paper is interesting from the point of view of disclosing the theoretical possibilities of the mechanization and automation of logical operations.

The paper of N. E. Kobrinskii and B. A. Trakhtenbrot, "On the construction of general theory of logical nets," may properly be considered one of the central papers in the logical-technological section of the collection. Here, the various aspects of designing logical nets are considered, a definition of the limited-deterministic operator is given, the physical realizability of any such operator is proven, the properties of these operators are adduced and the most general ways to synthesize them effectively are investigated. It is established, with this, that physical realization requires two types of elements: delay elements and logical elements, which, in the language of switching circuits, may be called one-cycle (nonsequential) circuits. The principles of organizing nets from standard elements of the types enumerated are considered, wherein the Sheffer stroke is taken as the basic logical element.

The paper contains theorems on the relationships between the operator weights and the number of lag elements. A hypothesis is asserted as to the decisive influence of the complexity of the formulas of Boolean algebra on the difficulty of the synthesis process and the complexity of the resulting design. For this, the complexity of a formula is estimated by its length. As one of the many considerations which enter into the estimate of a logical net, the authors choose the minimization requirement. Although significant minimization is impossible in the general case, one deals in practice with simple enough operators so that a significant amount of minimization is attainable.

The paper also considered the interpenetration of Boolean (logical) algebra and information theory. In conclusion, the paper cited the general course of the solution of the problems of synthesizing logical nets.

The character of the paper is such that an exact presentation of all the questions illuminated by it could turn it into an entire book. Therefore, the presentation is naturally sketchy in certain places, which in no way diminishes the value of the paper.

The paper of A. D. Kharkevich, "On switching circuits and their logical essence," is somewhat on the side of the traditional presentation of the applications of logic to technology. In it, the logical structure of switching circuits is considered from the engineering point of view. Discrete Q-poles are considered, the class of switching circuits is isolated among them and then the operations on these latter are investigated. It is possible that the logical principle of the paper could be distinguished more prominently.

In his paper, "Logical synthesis of switching circuits containing impedances and contacts," B. M. Rakov investigated the logical properties of two particular forms of four-terminal networks. Then, by means of Boolean algebra, these properties were used for the synthesis of circuits with impedances and contacts, making it possible to reduce significantly the number of the latter.

In his small work, "Analytical synthesis of sequential circuits from swiching formulas," A. N. Yurasov presented the essentials of this tableless logical-algebraic method which permits one, on a step-by-step basis, to eliminate the possible unrealizability of a primordial switching formula by the addition of intemediate elements.

The concluding paper of V. F. D'yachenko and V. G. Lazarev, "The use in telephony of logical algebra for the analysis and synthesis of switching circuits," is a purely engineering illustration of the capabilities of mathematical logic. On the basis of a concrete attempt to design a circuit in the Laboratory for Information Transmission Systems of the AN SSSR, the authors draw the conclusion that the mathematical methods of analyzing and synthesizing switching circuits which are based on the laws of logical algebra give a significant economization of time, decrease the influence of the designer's intuition and allow objective solutions to be obtained.

Thus, the technical papers of the collection on diverse problems show the importance of the technological applications of logic, and contain a new estimate of the results in this direction.

The definite many-sidedness of the second part enriches the collection. For example, it should be considered an obvious fact that an engineer or scientific worker acquainting himself with the technical applications of logic can only gain if, together with the "technicological" works, he becomes familiar with, say, the paper of Getmanova. This allows him to overcome the almost unavoidable limitations in questions of the foundations of mathematics, and to increase his familiarity with such questions.

While not having the intention of discussing the first part of the collection, the author may not deny himself the pleasure of naming the brillinat paper of E. Kol'man, "The significance of symbolic logic," the best paper in the collection.

The noteworthy fact, which it is necessary to hail, is the collaboration in one collection of philosophers and expounders of engineering and mathematics.

Together with the other collections on discrete automata and mathematical machines which appeared in 1958-1959, such as Problems of Cybernetics, Vol. 1, Questions in the Theory of Mathematical Machines, Vol. 1, Proceedings of the V. A. Steklov Institute of the AN SSSR, Vol. 51, and others, the present collection provides valuable material to the reader familiarizing himself or interested in instructing himself in the questions cited.

A fault which should be mentioned is the unwarranted delay in publishing the collection: from the autumn of 1956 to February 1959.

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